Exercices de Théorie de la Mesure

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Table des matières

1 Preliminaries

Sets

Let I be a non-empty set. Let $(A_i)_{i\in I}$ be a collection of sets. We recall that

$$\left(\bigcup_{i\in I} A_i\right)^c = \bigcap_{i\in I} A_i^c \qquad \left(\bigcap_{i\in I} A_i\right)^c = \bigcup_{i\in I} A_i^c.$$

If X and Y are two sets and $f : X \to Y$ a function, for any $B \subset Y$ we recall that

$$f^{-1}(B) := \{x \in X : f(x) \in B\}.$$

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Exercise 1.1. Let X, Y be two sets and $f : X \to Y$ be a function. For any $A, B, C \subset X$, any collection of subsets $(A_i)_{i\in I}$ of X and any collection $(B_i)_{i\in I}$ of subsets of Y, prove that

1. $f(A \cap B) \subset f(A) \cap f(B)$. Give an example for which the inclusion is strict ;
2. $f(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} f(A_i)$;
3. $f^{-1}(\bigcup_{i\in I} B_i) = \bigcup_{i\in I} f^{-1}(B_i)$;
4. $f^{-1}(\bigcap_{i\in I} B_i) = \bigcap_{i\in I} f^{-1}(B_i)$;
5. $f^{-1}(C^c) = f^{-1}(C)^c$.

Exercise 1.2. Let X, Y be two sets and $f : X \to Y$ be a function. Prove that for any $U \subset X$ and $V \subset Y$ we have

1. $f(f^{-1}(V)) \subset V$. Give an example for which the inclusion is strict ;
2. $U \subset f^{-1}(f(U))$. Give an example for which the inclusion is strict ;

Topological spaces

Let X be a set. A topology on X is a collection $\mathcal{T}_X$ of subsets of X such that

1. $\emptyset \in \mathcal{T}_X$ and $X \in \mathcal{T}_X$;
2. if $A, B \in \mathcal{T}_X$, then $A \cap B \in \mathcal{T}_X$;
3. if $U_i \in \mathcal{T}_X$ for all $i \in I$ for an index set I, then $\cup_{i\in I} U_i \in \mathcal{T}_X$.

The elements of $\mathcal{T}_X$ are called the open subsets of X. Closed subsets are the complements of open subsets : $F \subset X$ is said to be closed if $F^c \in \mathcal{T}_X$.

Let $x \in X$ and $A \subset X$. We say that

1. A is a neighbourhood of x if there exists an open subset $U \subset X$ (i.e. $U \in \mathcal{T}_X$) such that $x \in U$ and $U \subset A$;
2. x belongs to the closure of A, denoted $\bar{A}$, if for every open subset $U \subset X$ that contains x, we have $A \cap U \neq \emptyset$;

3. x belongs to the interior of A, denoted $\text{int}(A)$, if A is a neighbourhood of x ;
4. A is dense in X if for every non-empty $U \in \mathcal{T}_X$, we have $U \cap A \neq \emptyset$;

If $(Y, \mathcal{T}_Y)$ is another topological space, a map $f : X \to Y$ is said to be continuous if $f^{-1}(U) \in \mathcal{T}_X$, for any $U \in \mathcal{T}_Y$.

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Exercise 1.3. Let X be a topological space and $A, B \subset X$. Prove that

1. $(\text{int}A)^c = \overline{A^c}$;
2. $\overline{A}^c = \text{int}(A^c)$;
3. $\overline{A} = \overline{\overline{A}}$;
4. $\overline{A \cup B} = \overline{A} \cup \overline{B}$;
5. $\text{int}(\text{int}A) = \text{int}A$;
6. $\text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B)$;
7. $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$; give an example where the inclusion is strict ;
8. $\overline{\text{int}A} \subset \overline{A}$; give an example where the inclusion is strict ;

Exercise 1.4. Let X and Y be topological spaces and $f : X \to Y$ be a function. Then, f is continuous if and only if $f^{-1}(F)$ is closed in X for any closed subset $F \subset Y$.

Exercise 1.5. The Zariski topology τ_z on $\mathbb{R}$ is defined as follows : $A \in \tau_z \iff A = \emptyset$ or A^c is finite.

1. Which are the dense subsets of this topology ?
2. Which are the closed subsets with non-empty interior ?
3. What can be said about the identity $I : (\mathbb{R}, \tau_u) \to (\mathbb{R}, \tau_z)$, where τ_u is the usual topology on $\mathbb{R}$?

Metric spaces

Exercise 1.6. Let (X, d_X) and (Y, d_Y) be two metric spaces. Prove that a function $f : X \to Y$ is continuous in the classical ε-δ sense if and only if it is continuous in the topological sense for the topologies induced by the metrics.

Exercise 1.7. Let (X, d) be a metric space. Let $(x_n) \subset X$ be a sequence. Show that (x_n) converges to $x \in X$ if and only if any subsequence of (x_n) admits a further subsequence converging to x.

Exercise 1.8. Let $X = [0, 1]$ endowed with two distances d and D defined by $d(x, y) = |x - y|$ and $D(x, y) = |\sqrt{x} - \sqrt{y}|$ for any $x, y \in X$.

1. Show that $\tau_d = \tau_D$.
2. Does a $k > 0$ exist such that $D(x, y) \leq k\ d(x, y)$ for all $x, y \in X$?

Exercise 1.9. Let (X, d) be a metric space and $E \subset X$ be a subset. For any $x \in E$, the distance from x to E is defined by

$$d(x, E) := \inf\{d(x, y) : y \in E\}.$$

Show that the map $x \mapsto d(x, E)$ is continuous on X.

Vector spaces

Let $(X, \|.\|)$ be a normed vector space. A function $f : X \to \mathbb{R}$ is said to be coercive if

$$\lim_{\|x\| \to \infty} f(x) = +\infty,$$

that is for every $D > 0$, there exists $L > 0$ such that for all $x \in X$ with $\|x\| > L$, we have $f(x) > D$, i.e. for every sequence (x_n) with $\|x_n\| \to \infty$, we have $f(x_n) \to +\infty$.

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Exercise 1.10. Prove that the vector space $E := \mathcal{C}^0([0,1],\mathbb{R})$, endowed with the supremum norm $\|f\|_\infty := \sup_{x\in[0,1]} |f(x)|$ for any $f \in E$, is a Banach space.

Exercise 1.11. Let $E := (\mathcal{C}^0([0,1],\mathbb{R}), \|.\|_\infty)$.

1. Show that $A := \{f \in E : f(x) > 0 \ \forall\ x \in [0,1]\}$ is open ;
2. Show that $B := \{f \in E : \exists\ x \in [0,1], f(x) = 0\}$ is closed ;
3. Determine the border of $C := \{f \in E : f(0) > 0\}$;
4. Show that A is not open for the topology induced by the norm $\|f\|_1 := \int_0^1 |f(x)|\, dx$.

Exercise 1.12. Let $(X, \|.\|)$ be a normed vector space of finite-dimension. Let $f : X \to [0,\infty)$ be continuous and coercive. Show that f admits at least one global minimizer. Where does the finite-dimension hypothesis play a role ?

Exercise 1.13. Let $0 \le a_n \uparrow \infty$ be a non-decreasing sequence of non-negative real numbers.

1. If (c_n) is a sequence of real numbers converging to $c \in \mathbb{R}$, then
$$\frac{1}{a_n}\sum_{j=1}^{n-1}(a_{j+1} - a_j)c_j \to c.$$
Hint : first consider the case $c_n \to 0$.
2. If (x_n) is a sequence of real numbers such that the series $\sum_{j=1}^{\infty} x_j/a_j$ converges in $\mathbb{R}$, then we have
$$\frac{1}{a_n}\sum_{j=1}^{n} x_j \to 0.$$
Hint : set $c_0 = 0$, $c_n = \sum_{j=1}^{n} x_j/a_j$ and use the previous point.
This result is known as "Kronecker's lemma".

Exercise 1.14. Let $f : (a,b) \to \mathbb{R}$ be differentiable on (a,b). Prove that f is Lipschitz on (a,b) if and only if f' is bounded on (a,b).

The German mathematician Hans Rademacher $(1892 - 1969)$ proved that if a function $f : U \to \mathbb{R}$, defined on an open subset $U \subset \mathbb{R}^n$, is Lipschitz, then the set of points at wich f is not differentiable has Lebesgue measure 0.

Exercise 1.15. Show that $|\sqrt{x} - \sqrt{y}| \le \sqrt{|x-y|}$ for every $x, y \ge 0$. Conclude that $x \mapsto \sqrt{x}$ is uniformly continuous on $[0,\infty)$. Is it Lipschitz on some compact subset of $[0,\infty)$? Is it Lipschitz on $[0,\infty)$?

2 Classes of subsets

Definition. Let Ω be a set and $\mathcal{A}$ be a non-empty class of subsets of Ω. Then we say that $\mathcal{A}$ is a

1. π-system if $A \cap B \in \mathcal{A}$ for any $A, B \in \mathcal{A}$;
2. semiring if $A \cap B \in \mathcal{A}$ for any $A, B \in \mathcal{A}$, and if for any $A, B \in \mathcal{A}$ with $A \subset B$ there exists $m \geq 1$ and $C_1, \ldots, C_m \in \mathcal{A}$ pairwise disjoint such that $B \setminus A = \biguplus_{j=1}^{m} C_j$;
3. ring if $A \cup B \in \mathcal{A}$ and $A \setminus B \in \mathcal{A}$ for any $A, B \in \mathcal{A}$;
4. algebra if $\emptyset \in \mathcal{A}$, $A^c \in \mathcal{A}$ for any $A \in \mathcal{A}$, and $A \cup B \in \mathcal{A}$ for any $A, B \in \mathcal{A}$;
5. σ-algebra (or σ-field) if $\emptyset \in \mathcal{A}$, $A^c \in \mathcal{A}$ for any $A \in \mathcal{A}$, and $\cup_{n\in\mathbb{N}} A_n \in \mathcal{A}$ for any sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$.

Observe that if $\mathcal{A}$ is an algebra, then $A \cap B \in \mathcal{A}$ whenever $A, B \in \mathcal{A}$ since $A \cap B = (A^c \cup B^c)^c$.

The algebra generated by some set of subsets $\mathcal{F} \subset 2^{\Omega}$ is denoted $f(\mathcal{F})$ and the σ-algebra generated by $\mathcal{F}$ is written $\sigma(\mathcal{F})$, where 2^{Ω} denotes the set of all subsets of Ω.

On a topological space X, the σ-field generated by the open subsets of X is called the Borel σ-field. The Borel σ-field of $\mathbb{R}^d$ will be denoted $\mathcal{R}^d$ throughout these notes.

For any $A \in \mathcal{R}^d$, we also let

$$\mathcal{R}^d_A := \{A \cap B : B \in \mathcal{R}^d\}.$$

A subset S of a topological space X is said to be a Lindelof set if every open cover of S admits a countable subcover. By definition, compact subsets are Lindelof. When X is a metric space, one can show that a subset S of X is Lindelof if and only if S is separable, i.e. admits a countable and dense subset. Therefore, open subsets of $\mathbb{R}^n$, $n \geq 1$, are Lindelof.

The symbol $\uplus$ denotes the classical union symbol $\cup$ when all sets involved are pairwise disjoint.

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Exercise 2.1. Justify that if $\mathcal{F}$ is a semiring or a ring of some space X, then $\emptyset \in \mathcal{F}$.

Exercise 2.2. Prove that a class $\mathcal{A}$ of subsets of X is an algebra if and only if $X \in \mathcal{A}$ and $A \setminus B \in \mathcal{A}$ whenever $A, B \in \mathcal{A}$.

Exercise 2.3. For any $A \in \mathcal{R}^d$, prove that

$$\mathcal{R}^d_A = \{B \in \mathcal{R}^d : B \subset A\}\cdot$$

Exercise 2.4. Let $\mathcal{F}$ be a semiring of some space Ω. Show that for any $A, B \in \mathcal{F}$, there exists $m \geq 1$ and $C_1, \ldots, C_m \in \mathcal{F}$ pairwise disjoint such that $B \setminus A = \biguplus_{j=1}^{m} C_j$.

Exercise 2.5. Let Ω be a finite set. Prove that the cardinality of 2^{Ω} is exactly $2^{\#\Omega}$.

Exercise 2.6. Let $\Omega = \{1, 2, 3\}$ and $\mathcal{F} = \Big\{\{1\}, \{2\}, \{3\}\Big\}$ be a class of subsets of Ω. Compute explicitly $\sigma(\mathcal{F})$, $f(\mathcal{F})$ and 2^{Ω}.

Exercise 2.7. Let Ω be a space. Let $\mathcal{F}$ be a set of subsets of Ω. Prove that

1. if $\mathcal{F}$ is an algebra, then $\mathcal{F}$ is a ring ;
2. if $\mathcal{F}$ is a ring, then $\mathcal{F}$ is a semi-ring ;
3. if $\mathcal{F}$ is a semiring, then $\mathcal{F}$ is a π-system ;
4. if $\mathcal{F}$ is a semi-ring, then

$$\mathcal{F}^+ := \left\{ B \subset \Omega : \exists\, m \geq 1,\ A_1, \ldots, A_m \in \mathcal{F} \text{ disjoint, such that } B = \biguplus_{j=1}^{m} A_j \right\}$$

is a ring, and actually the smallest ring that contains $\mathcal{F}$.
HINT : prove that $\mathcal{F}^+$ is a π-system, is stable by finite disjoint unions and then write $A \cup B$ as a disjoint union.

Exercise 2.8. Show that the set $\mathcal{A} := \{(a, b] : -\infty < a \leq b < +\infty\}$ is a semiring of $\mathbb{R}$.

Exercise 2.9. Prove that if $\mathcal{F}_i$ is a semiring on Ω_i, $i = 1, 2$, then $\mathcal{F}_1 \times \mathcal{F}_2 := \{A_1 \times A_2 : A_i \in \mathcal{F}_i\}$ is a semi-ring on $\Omega_1 \times \Omega_2$. Show that $\mathcal{F}_1 \times \mathcal{F}_2$ may not be a ring even if $\mathcal{F}_1$ and $\mathcal{F}_2$ are algebras.

Exercise 2.10. Let Ω be a set and $\mathcal{A} = \{A \subset \Omega : A \text{ or } A^c \text{ is at most countable}\}$. Prove that $\mathcal{A}$ is a σ-algebra.

Exercise 2.11. Let $\mathcal{A} = \{A \subset \mathbb{N} : A \text{ or } A^c \text{ is finite}\}$. Prove that $\mathcal{A}$ is an algebra but not a σ-field.

Exercise 2.12. Let $\{\mathcal{F}_i\}_{i \in I}$ be an arbitrary collection of (σ-)algebras of a same space. Prove that $\bigcap_{i \in I} \mathcal{F}_i$ is a (σ-)algebra.

Exercise 2.13. Let $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3 \subset \ldots$ be an increasing sequence of algrebas. Prove that $\bigcup_{j=1}^{\infty} \mathcal{F}_j$ is an algebra.

The result is no longer true for σ-fields. In fact, Broughton & Huff proved the following in "A comment on unions of sigma-fields" (1977) : if the σ-fields $\mathcal{F}_n$ are such that, for all $n \geq 1$, $\mathcal{F}_n$ is strictly contained in $\mathcal{F}_{n+1}$, then $\bigcup_{n=1}^{\infty} \mathcal{F}_n$ is not a σ-field.

Exercise 2.14. Let X be a space. Show that the collection of all single-point sets of X generates the σ-algebra of Exercice 2.10.

Exercise 2.15. Let Ω be a finite set of cardinality $\#\Omega = 2p$, $p \in \mathbb{N}$. Consider the collection

$$K := \{A \subset \Omega : \#A = 2r \text{ for some } r \in \{0, \ldots, p\}\}.$$

1. Show that $\emptyset \in K$, K is stable by complements and by countable disjoint unions ;
2. For which values of p is K an algebra and/or a σ-field ?

Exercise 2.16. Let Ω be a set and $\mathcal{A} \subset 2^{\Omega}$ be some set of subsets of Ω. Show that for any $B \in \sigma(\mathcal{A})$, there exists an at most countable subset $\mathcal{A}_B \subset \mathcal{A}$ such that $B \in \sigma(\mathcal{A}_B)$.

HINT : consider $\mathcal{F} := \{B \in \sigma(\mathcal{A}) : \exists\, \mathcal{A}_B \subset \mathcal{A} \text{ such that } \#\mathcal{A}_B \leq \#\mathbb{N} \text{ and } B \in \sigma(\mathcal{A}_B)\}$.

Exercise 2.17. Let Ω be a set and $A_1, \ldots, A_n$ be subsets of Ω. For all $\alpha \in \{0,1\}^n$, let $F_n(\alpha) := \bigcap_{j=1}^{n} A_i^{\alpha_i}$, where $A_i^0 = A_i$ and $A_i^1 = A_i^c$.

1. Prove that the collection $\{F_n(\alpha) : \alpha \in \{0,1\}^n\}$ is a partition of Ω ;
2. Show that $f(\{A_1, \ldots, A_n\}) = E := \left\{ \biguplus_{\alpha \in J} F_n(\alpha) : J \subset \{0,1\}^n \right\}$;
3. Deduce that $\# f(\{A_1, \ldots, A_n\}) \leq 2^{2^n}$;
4. Given a countable collection $(A_n)_n$ of subsets of Ω, conclude that $f(\{A_n\}_{n=1}^{\infty})$ is at most countable ;
   HINT : Show that $f(\{A_n\}_{n=1}^{\infty}) = \bigcup_{n \geq 1} f(\{A_1, \ldots, A_n\})$.
5. Is it true that $\sigma(\{A_n\}_{n=1}^{\infty})$ is always at most countable ?

Exercise 2.18. Let $(A_n)_n$ be a partition of some space Ω. Show that

$$\sigma((A_n)_n) = \left\{ \biguplus_{j \in J} A_j : J \subset \mathbb{N} \right\}.$$

What is the cardinality of $\sigma((A_n)_n)$ in this case ?

Exercise 2.19. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of subsets of some space Ω. Considering the construction of Exercice 2.17, show that $\sigma((A_n)_n)$ is either finite or has cardinality at least that of the continuum (i.e. the cardinality of $\mathbb{R}$). To do so, proceed as follows. Let us define, in a similar way as in Exercice 2.17, for any $\alpha \in \{0,1\}^{\mathbb{N}}$,

$$F(\alpha) := \bigcap_{n\in\mathbb{N}} A_n^{\alpha_n}.$$

1. Show that $\{F(\alpha) : \alpha \in \{0,1\}^{\mathbb{N}}\}$ is a partition of Ω ;
2. Let us define the auxiliary sets

$$E_1 := \left\{ \biguplus_{\alpha\in J} F(\alpha) : J \subset \{0,1\}^{\mathbb{N}},\ J \text{ is a most countable} \right\},$$

$$E_2 := \left\{ \biguplus_{\alpha\in J} F(\alpha) : J \subset \{0,1\}^{\mathbb{N}} \right\}.$$

   Prove that $E_1 \subset \sigma((A_n)_n) \subset E_2$.
3. Prove the statement.
   HINT : let $G := \{\alpha \in \{0,1\}^{\mathbb{N}} : F(\alpha) \neq \emptyset\}$. Consider the cases $\#G < \infty$ and $\#G = \infty$ seperately.

Exercise 2.20. Let X and Y be spaces. Let $\mathcal{A} \subset 2^X$ and $\mathcal{B} \subset 2^Y$ be σ-algebras of X and Y respectively. Let $f : X \to Y$ be an arbitrary mapping. Then,

1. the class $f^{-1}(\mathcal{B}) := \{f^{-1}(B) : B \in \mathcal{B}\}$ is a σ-algebra of X, called the σ-field generated by f ;
2. the class $\{B \subset Y : f^{-1}(B) \in \mathcal{A}\}$ is a σ-algebra of Y ;
3. for any class $\mathcal{F}$ of subsets of Y, we have $\sigma(f^{-1}(\mathcal{F})) = f^{-1}(\sigma(\mathcal{F}))$.

Point 1 has practical implications. Let X be a set and $f : X \to \mathbb{R}$ be a map. The function f allows one to equip X with the structure of a measure space, i.e. define a σ-algebra on X by setting $\mathcal{F} := \{f^{-1}(A) : A \in \mathcal{B}\}$, where $\mathcal{B}$ is the Borel σ-algebra of $\mathbb{R}$.

Exercise 2.21. Consider the Borel σ-algebra $\mathcal{R}^d$ of $\mathbb{R}^d$, i.e. $\mathcal{R}^d = \sigma(\mathcal{U})$ where $\mathcal{U}$ is the set of open subsets of $\mathbb{R}^d$. Show that $\mathcal{R}^d$ is either generated by

0. $\mathcal{F}_0$ = open balls ;
   HINT : use the Lindelof property of open sets.
1. $\mathcal{F}_1$ = open balls with rational radius ;
2. $\mathcal{F}_2$ = open balls with rational radius and center point with rational coordinates.
   HINT : justify that

$$\sigma(\mathcal{F}_2) \subset \sigma(\mathcal{F}_1) \subset \sigma(\mathcal{F}_0) \subset \mathcal{R}^d.$$

   Then prove that $\mathcal{U} \subset \sigma(\mathcal{F}_0)$, $\mathcal{F}_0 \subset \sigma(\mathcal{F}_1)$, $\mathcal{F}_1 \subset \sigma(\mathcal{F}_2)$. To prove $\mathcal{F}_1 \subset \sigma(\mathcal{F}_2)$, let $B(x,r) \in \mathcal{F}_1$, $r \in \mathbb{Q}$ and let $(x_n) \subset \mathbb{Q}^d$ be such that $|x_n - x| < 1/n$. Prove that $B(x,r] = \bigcap_{n\geq 1} B(x_n, r + 1/n) \in \sigma(\mathcal{F}_2)$ and then approximate $B(x,r)$ from inside with closed balls.
3. $\mathcal{F}_3$ = closed subsets ;
4. $\mathcal{F}_4$ = closed balls ;
5. $\mathcal{F}_5 = \{\prod_{i=1}^d (a_i, b_i) : a_i, b_i \in \mathbb{R}\}$; $\mathcal{F}_5{}' = \{\prod_{i=1}^d (a_i, b_i) : a_i, b_i \in \mathbb{Q}\}$;
   HINT : it is obvious that $\mathcal{F}_5 \subset \mathcal{U}$. Then prove that $\mathcal{F}_0 \subset \sigma(\mathcal{F}_5)$.
6. $\mathcal{F}_6 = \{\prod_{i=1}^d [a_i, b_i] : a_i, b_i \in \mathbb{R}\}$; $\mathcal{F}_6{}' = \{\prod_{i=1}^d [a_i, b_i] : a_i, b_i \in \mathbb{Q}\}$;
7. $\mathcal{F}_7 = \{\prod_{i=1}^d [a_i, b_i) : a_i, b_i \in \mathbb{R}\}$; $\mathcal{F}_7{}' = \{\prod_{i=1}^d [a_i, b_i) : a_i, b_i \in \mathbb{Q}\}$;
8. $\mathcal{F}_8 = \{\prod_{i=1}^d (a_i, b_i] : a_i, b_i \in \mathbb{R}\}$; $\mathcal{F}_8{}' = \{\prod_{i=1}^d (a_i, b_i] : a_i, b_i \in \mathbb{Q}\}$;

3 Set functions and measures

Definition. Let Ω be a set and $\mathcal{A}$ be a class of subsets of Ω. A set function $\mu : \mathcal{A} \to (-\infty, \infty]$ is called

1. additive if $\mu(\uplus_{i=1}^n A_i) = \sum_{i=1}^n \mu(A_i)$ for any $n \geq 1$ and any disjoint sets $A_1, \ldots, A_n \in \mathcal{A}$ such that $\bigcup_{i=1}^n A_i \in \mathcal{A}$;
2. subadditive if $\mu(\cup_{i=1}^n A_i) \leq \sum_{i=1}^n \mu(A_i)$ for any $n \geq 1$ and any $A_i \in \mathcal{A}$ with $\bigcup_{i=1}^n A_i \in \mathcal{A}$;
3. countably additive (or σ-additive) if $\mu(\uplus_{n=1}^\infty A_n) = \sum_{n=1}^\infty \mu(A_n)$ for any pairwise disjoint sets $A_n \in \mathcal{A}$ such that $\biguplus_{n=1}^\infty A_n \in \mathcal{A}$. One may similarly define countably subadditive set functions.

Definition. A σ-additive set function $\mu : \mathcal{A} \to [0, \infty]$ such that $\emptyset \in \mathcal{A}$ and $\mu(\emptyset) = 0$ is called a measure on $\mathcal{A}$. If $\Omega \in \mathcal{A}$ and $\mu(\Omega) = 1$, μ is called a probability measure. The triple $(\Omega, \mathcal{A}, \mu)$ is called a measure space if $\mathcal{A}$ is a σ-algebra of sets of Ω and if μ is a measure on $\mathcal{A}$.

If $(\Omega, \mathcal{A}, \mu)$ is a measure space and $\mathcal{F} \subset \mathcal{A}$ is a subclass, we say that μ is σ-finite on $\mathcal{F}$ if there exists a sequence $A_n \in \mathcal{F}$, $n \geq 1$ such that $\Omega = \bigcup_{n=1}^\infty A_n$ with $\mu(A_n) < \infty$ for any n.

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Exercise 3.1. Prove that an additive set function $\mu : \mathcal{A} \to [0, \infty]$ defined on a ring $\mathcal{A}$ is subadditive. Can you say where the hypothesis that $\mathcal{A}$ is a ring plays a role?

Exercise 3.2. Let μ be a measure on a ring $\mathcal{F}$. Prove that the class of all sets $Z \in \mathcal{F}$ of measure 0 is a ring.

Exercise 3.3. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. Show that $\mathcal{A} := \{A \in \mathcal{F} : \mu(A) \in \{0, 1\}\}$ is a σ-field.

Exercise 3.4. Let $\mathcal{A}$ be the σ-algebra of Exercice 2.10 on a space Ω.

1. Assume that Ω is uncountable. For any at most countable $A \in \mathcal{A}$, let $\mu(A) = 0$, and for A with at most countable complement let $\mu(A) = 1$. Prove that μ is well-defined and is a measure on $\mathcal{A}$.
2. Show that μ is not well-defined if Ω is at most countable,

Exercise 3.5. Let $\mathcal{A}$ be the algebra of Exercice 2.11. For finite $A \in \mathcal{A}$, let $\mu(A) = 0$, and for A with finite complement let $\mu(A) = 1$. Prove that μ is additive but not countably additive on $\mathcal{A}$.

Exercise 3.6. Consider the σ-field 2^Ω on Ω, and for all $A \in 2^\Omega$, let $\mu(A) = \#A$ if $\#A < \infty$ and let $\mu(A) = \infty$ if A is infinite. Show that μ is a measure. It is called the counting measure. When is μ a finite measure? When is it σ-finite?

Exercise 3.7. Let Ω be a space and $\mathcal{F}$ be a σ-field on Ω that contains all singletons. Let $x \in \Omega$ and let $\delta_x(A) := 1$ if $x \in A$ and $\delta_x(A) = 0$ if $x \notin A$, for every $A \in \mathcal{F}$. In short, $\delta_x(A) = 1_{\{x \in A\}}$. Show that δ_x is a measure on $\mathcal{F}$. It is called Dirac's measure. If μ is another measure on $\mathcal{F}$ such that $\mu(A) = 0$ for all $A \in \mathcal{F}$ that do not contain x, prove that there exists $C \in [0, +\infty]$ such that $\mu = C\delta_x$ on $\mathcal{F}$.

Exercise 3.8. Let $(X, \mathcal{A}, \mu)$ be a measure space and (μ_n) a non-decreasing sequence of measures, i.e. for every n we have $\mu_n(A) \le \mu_{n+1}(A)$ for all $A \in \mathcal{A}$. Let us define $\mu(A) := \lim_{n\to\infty} \mu_n(A)$ for every $A \in \mathcal{A}$. Show that μ is a measure on $\mathcal{A}$.

Exercise 3.9. Let $(X, \mathcal{A}, \mu)$ be a measure space and (μ_n) be a sequence of measures. We define $\mu(A) := \sum_{n\ge1} \mu_n(A)$ for every $A \in \mathcal{A}$. Is μ a measure on $\mathcal{A}$?

Exercise 3.10. On $(\mathbb{N}, 2^{\mathbb{N}})$ let $\mu_n(A) := \#(A \cap [n, +\infty))$ for any $A \subset \mathbb{N}$, $n \ge 1$. Show that (μ_n) is a non-increasing sequence of measures. Let $\mu(A) := \lim_{n\to\infty} \mu_n(A)$ for any $A \subset \mathbb{N}$. Is μ a measure on $2^{\mathbb{N}}$? Compute $\mu(\mathbb{N})$ and $\mu(\{k\})$ for every $k \in \mathbb{N}$.

Exercise 3.11. Let μ be a measure on a ring $\mathcal{F}$.

1. Let $A, B \in \mathcal{F}$ be such that either $\mu(A) < \infty$ or $\mu(B) < \infty$. Show that
$$\mu(A) - \mu(B) \le \mu(A \setminus B).$$
2. Let $(A_n)_n, (B_n)_n \subset \mathcal{F}$ be such that $\bigcup_{n\ge1} A_n, \bigcup_{n\ge1} B_n \in \mathcal{F}$. Assume that $\mu(\bigcup_{n\ge1} A_n) < \infty$ or $\mu(\bigcup_{n\ge1} B_n) < \infty$. Prove that
$$\mu\Big(\bigcup_{n\ge1} A_n\Big) - \mu\Big(\bigcup_{n\ge1} B_n\Big) \le \sum_{n\ge1} \mu\Big(A_n \setminus \Big(\bigcup_{k\ge1} B_k\Big)\Big) \le \sum_{n\ge1} \mu(A_n \setminus B_n).$$

Exercise 3.12. Let $\mathcal{A}$ be a ring of sets and $\mu : \mathcal{A} \to [0, \infty]$ be a set function such that $\mu(\emptyset) = 0$. Show that μ is countably additive if and only if μ is additive and countably subadditive.

Exercise 3.13. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. The lim sup and lim inf of a sequence of subsets $(C_n)_n \subset \mathcal{F}$ are defined as follows,
$$\liminf_{n\ge1} C_n = \bigcup_{n\ge1} \bigcap_{k\ge n} C_k$$
$$\limsup_{n\ge1} C_n = \bigcap_{n\ge1} \bigcup_{k\ge n} C_k.$$
Given two sequences $(A_n)_n, (B_n)_n \subset \mathcal{F}$ show from these definitions that

1. $\liminf A_n \subset \limsup A_n$;
2. $\Big(\limsup A_n\Big)^c = \liminf A_n^c$;
3. $\limsup(A_n \cap B_n) \subset \Big(\limsup A_n\Big) \cap \Big(\limsup B_n\Big)$;
4. $\limsup(A_n \cup B_n) = \Big(\limsup A_n\Big) \cup \Big(\limsup B_n\Big)$;
5. $\liminf(A_n \cup B_n) \subset \Big(\liminf A_n\Big) \cup \Big(\liminf B_n\Big)$;
6. $\liminf(A_n \cap B_n) = \Big(\liminf A_n\Big) \cap \Big(\liminf B_n\Big)$;
7. If $A_n \to A$ (that is, $\liminf A_n = A = \limsup A_n$) and $B_n \to B$, then, $A_n \cup B_n \to A \cup B$ and $A_n \cap B_n \to A \cap B$;
8. $\mu(\liminf_{n\ge1} A_n) \le \liminf_{n\to\infty} \mu(A_n)$. What can be said about $\mu(\limsup_{n\ge1} A_n)$?

Exercise 3.14. The goal of this exercice is to construct a non-measurable set following Vitali's example. Consider the relation $\sim$ on $[0, 1)$: $x \sim y$ if and only if $x - y \in \mathbb{Q}$.

1. Show that $\sim$ is an equivalence relation ;
2. Consider the quotient space $\mathcal{A} := [0, 1)/\sim$. By the axiom of choice, let us select exactly one element in each equivalence class, i.e. there exists a map $\psi : \mathcal{A} \to [0, 1)$ such that $\psi(\alpha) \in \alpha$ for every $\alpha \in \mathcal{A}$. Set $B = \psi(\mathcal{A})$, i.e. the collection of all chosen representatives. Show that
$$\{B + q : q \in \mathbb{Q}, |q| < 1\}$$
is a familiy of disjoint subsets whose union contains $[0, 1)$ and is contained in $[-1, 2]$.

3. Show that the measurability of B would lead to a contradiction.

Exercise 3.15. The goal of this exercice is to construct the triadic Cantor set, which is uncountable, compact and has Lebesgue measure 0. The triadic Cantor's set is the subset of $[0,1]$ that remains after removing successively the interval $(1/3, 2/3)$, then the intervals $(1/3^2, 2/3^2)$ and $(7/3^2, 8/3^2)$, then the four intervals

$$\left(\frac{1}{3^3}, \frac{2}{3^3}\right), \left(\frac{7}{3^3}, \frac{8}{3^3}\right), \left(\frac{19}{3^3}, \frac{20}{3^3}\right), \left(\frac{25}{3^3}, \frac{26}{3^3}\right),$$

and so on. At each step, the middle open intervals of the remaining intervals is removed. This process can be described as follows. Let $C_0 = [0,1]$. At each step n, the remaing subset is

$$C_n = \frac{1}{3}C_{n-1} \biguplus \left(\frac{2}{3} + \frac{1}{3}C_{n-1}\right) \subset C_{n-1}, \ n \geq 1.$$

The triadic Cantor set can therefore be defined as $\mathcal{C} := \bigcap_{n\geq 1} C_n$.

1. Justify that $\mathcal{C}$ is compact ;
2. Prove that $\mathcal{C}$ has Lebesgue measure 0 ;
3. For any $x \in [0,1)$, define the sequence $(\alpha_n)_n \subset \mathbb{N}$ as follows : $\alpha_1 := \lfloor 3x \rfloor$ and

$$\alpha_n := \left\lfloor 3^n \left(x - \sum_{k=1}^{n-1} \alpha_k 3^{-k}\right)\right\rfloor, \quad n \geq 2.$$

   Prove that $\alpha_n \in \{0,1,2\}$ for any $n \geq 1$.
4. Let $x \in [0,1)$ and $(\alpha_n)_n$ be the associated sequence defined in 3. Prove that for any $n \geq 1$, we have

$$\sum_{k=1}^{n} \frac{\alpha_k}{3^k} \leq x < \sum_{k=1}^{n} \frac{\alpha_k}{3^k} + \frac{1}{3^n},$$

   and conclude that

$$x = \sum_{n=1}^{\infty} \frac{\alpha_n}{3^n}.$$

   Prove that the sequence $(\alpha_n)_n$ is such that there is no $n_0 \geq 1$ such that $\alpha_n = 2$ for all $n \geq n_0$. We write $x = 0 \cdot \alpha_1\alpha_2 \ldots$; this is called the triadic representation of x.

   Let us denote by $\mathcal{N}$ the space of sequences $(\alpha_n)_n$ in $\{0,1,2\}$ that are not eventually equal to 2.
5. When constructing $\mathcal{C}$, step 1 consists in dividing $[0,1)$ into 3 disjoint intervals ($I_0 = [0,1/3), I_1 = [1/3, 2/3), I_2 = [2/3, 1)$, say), step 2 in dividing each I_i into 3 disjoint intervals again ($I_{i,0}, I_{i,1}, I_{i,2}$, say) and so on. At each step n, we have divided $[0,1)$ into 3^n disjoint intervals $I_{i_1,i_2,\ldots,i_n}$, $i_j \in \{0,1,2\}$, $j = 1,\ldots,n$ of the form $[a,b)$. Prove that

$$I_{i_1,\ldots i_n} = \left[\sum_{k=1}^{n} \frac{i_k}{3^k} \ ; \ \sum_{k=1}^{n} \frac{i_k}{3^k} + \frac{1}{3^n}\right).$$

   Show that for any $x = 0 \cdot \alpha_1\alpha_2 \ldots \in [0,1)$, $(\alpha_n) \in \mathcal{N}$, and any $n \geq 1$, we have that $x \in I_{i_1,\ldots i_n}$ if and only if $i_k = \alpha_k$ for every $k = 1,\ldots,n$. Conclude that for any $x \in [0,1)$, the decomposition $x = 0 \cdot \alpha_1\alpha_2 \ldots$ is unique in $\mathcal{N}$.

   In other words, the coefficients $(\alpha_n)_n$ tell exactly in which intervals of the decomposition of $[0,1)$ x lies.
6. Prove that if $x = 0 \cdot \alpha_1\alpha_2 \ldots \in [0,1)$, $(\alpha_n)_n \in \mathcal{N}$, is removed from $[0,1)$ at some point in the construction of $\mathcal{C}$, there exists $k \geq 1$ such that $\alpha_k = 1$. Conclude that $\mathcal{C}$ is uncountable.

4 Carathéodory's extension theorem

Definition. A measure space $(X, \mathcal{A}, \mu)$ is said to be complete if for any $A \in \mathcal{A}$ with $\mu(A) = 0$ and any $B \subset A$, we have $B \in \mathcal{A}$.

Proposition (Completion of measures)**.** Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let

$$\mathcal{F}_\mu := \Big\{ A \subset \Omega : A = A' \cup N \text{ with } A' \in \mathcal{F} \text{ and } N \subset \tilde{N}, \tilde{N} \in \mathcal{F}, \mu(\tilde{N}) = 0 \Big\}.$$

For any $A := A' \cup N \in \mathcal{F}_\mu$, where $A' \in \mathcal{F}$, $N \subset \tilde{N}$, $\tilde{N} \in \mathcal{F}$ and $\mu(\tilde{N}) = 0$, let us define $\bar{\mu}(A) := \mu(A')$. Then $(\Omega, \mathcal{F}_\mu, \bar{\mu})$ is a complete measure space such that $\bar{\mu}_{|\mathcal{F}} = \mu$.

Note that $\mathcal{F}_\mu$ is the smallest σ-field for which an extension of μ is complete : if $\tilde{\mathcal{F}}$ is a σ-field with $\mathcal{F} \subset \tilde{\mathcal{F}}$, and if $\tilde{\mu}$ is a measure on $\tilde{\mathcal{F}}$ such that $(\Omega, \tilde{\mathcal{F}}, \tilde{\mu})$ is complete and $\tilde{\mu}_{|\mathcal{F}} = \mu$, then $\mathcal{F}_\mu \subset \tilde{\mathcal{F}}$.

Theorem (Extension theorem)**.** Let Ω be a non empty-set, $\mathcal{A} \subset 2^\Omega$ be a semi-ring and $\mu : \mathcal{A} \to [0, \infty]$ be a set function. Assume that $\mu(\emptyset) = 0$, μ is additive and σ-subadditive. Let us define the set function $\mu^* : 2^\Omega \to [0, \infty]$, also called *outer-measure*, given by

$$\mu^*(A) := \inf \left\{ \sum_{n=1}^{\infty} \mu(A_n) : A_n \in \mathcal{A},\ A \subset \bigcup_{n=1}^{\infty} A_n \right\}, \quad \forall\, A \subset \Omega.$$

Let

$$\mathcal{M}(\mu^*) := \Big\{ A \subset \Omega : \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A),\ \forall\, E \subset \Omega \Big\}.$$

Then the following holds :

1. $\mathcal{M}(\mu^*)$ is a σ-field that contains $\sigma(\mathcal{A})$;
2. The restriction of μ^* to $\mathcal{M}(\mu^*)$ is a measure on $\mathcal{M}(\mu^*)$ and $\mu^*_{|\mathcal{A}} = \mu$;
3. $(\Omega, \mathcal{M}(\mu^*), \mu^*_{|\mathcal{M}(\mu^*)})$ is a complete measure space. In particular, we have
$$\mathcal{A} \subset \sigma(\mathcal{A}) \subset \sigma(\mathcal{A})_\mu \subset \mathcal{M}(\mu^*);$$
4. $\mu^*_{|\sigma(\mathcal{A})_\mu} = \overline{\mu^*_{|\sigma(\mathcal{A})}}$, i.e. the restriction of μ^* to $\sigma(\mathcal{A})_\mu$ coincides with the completion of $\mu^*_{|\sigma(\mathcal{A})}$;
5. If μ is σ-finite on $\mathcal{A}$, then $\mathcal{M}(\mu^*) = \sigma(\mathcal{A})_\mu$.

To construct the Lebesgue measure on $\mathbb{R}$, an additive and σ-subadditive set function λ is first defined on the semiring $\mathcal{A} := \big\{]a, b] : -\infty < a \leq b < \infty \big\}$, by letting $\lambda(]a, b]) = (b - a)$. The outer measure λ^* associated to λ is a measure when restricted to $\mathcal{M}(\lambda^*)$. As previously mentioned, $\mathcal{M}(\lambda^*)$ contains $\sigma(\mathcal{A}) = \mathcal{R}$, the Borel subsets of $\mathbb{R}$, and we further have that $\mathcal{M}(\lambda^*) = \mathcal{R}_{\lambda^*}$. The elements of $\mathcal{M}(\lambda^*)$ are called the Lebesgue measurable sets of $\mathbb{R}$ and $\lambda^*_{|\mathcal{M}(\lambda^*)}$ is called the Lebesgue measure on $\mathbb{R}$. For the sake of convenience, we will always denote the Lebesgue measure by λ.

Theorem (Uniqueness theorem)**.** Let $\mathcal{P}$ be a π-system and μ_1, μ_2 be two measures on $\sigma(\mathcal{P})$, σ-finite on $\mathcal{P}$. If $\mu_1 = \mu_2$ on $\mathcal{P}$, then $\mu_1 = \mu_2$ on $\sigma(\mathcal{P})$.

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Exercise 4.1. Let λ denote the Lebesgue measure on $\mathbb{R}$ defined on the Lebesgue measurable subsets of $\mathbb{R}$. The goal of this exercice is to show that translations of Lebesgue measurable sets are Lebesgue measurable sets and that λ is invariant under translations.

1. Prove that if B is a Borel subset of $\mathbb{R}$, then $B+x$ is also a Borel subset of $\mathbb{R}$ for every $x \in \mathbb{R}$.
   HINT : let $x \in \mathbb{R}$ and look at $\mathcal{F} := \{B \in \mathcal{R} : B + x \in \mathcal{R}\}$.
2. Prove that if $B \in \mathcal{R}$ and $x \in \mathbb{R}$, then $\lambda(B + x) = \lambda(B)$.
   HINT : look at the measure $\nu(B) := \lambda(B + x)$, $B \in \mathcal{R}$ defined on the Borel subsets (first show that this is indeed a measure on $\mathcal{R}$) and use the uniqueness theorem. Can you think of another way to show translation invariance of λ on $\mathcal{R}$?
3. Using the fact that $\mathcal{M}(\lambda^*) = \mathcal{R}_{\lambda^*}$, prove that if A is a Lebesgue measurable subset of $\mathbb{R}$, then $A + x$ is also Lebesgue measurable for any $x \in \mathbb{R}$, and show that $\lambda(A + x) = \lambda(A)$.

Exercise 4.2. Let $(I =]0, 1], \mathcal{R}_I, \mu)$ be a probability measure defined on the Borel subsets of $]0, 1]$. Assume that $\mu(A) = \mu(B)$ for any Borel sets $A, B \subset I$ that are translations of one another. Show that μ coincides with the Lebesgue measure on I. To do so, proceed as follows :

1. Show that μ coincide with λ on sets of the form $C_j :=]0, 1/j]$, where $j \in \mathbb{N}$ and $j \geq 1$.
2. Show that μ coincide with λ on sets of the form $D_{j,k} :=]0, k/j]$, where $j, k \in \mathbb{N}$, $j \geq 1$, $k \in \{0, \ldots, j-1\}$. Deduce that μ and λ coincide on every set of the form $]a, b]$ with $-\infty < a < b < \infty$ and $a, b \in \mathbb{Q}$.
3. Conclude that $\mu = \lambda$ on $\mathcal{R}_I$.

Exercise 4.3. Let $\mathcal{A}$ be a semiring of some space Ω. Let $\mathcal{A}^+$ be the smallest ring containing $\mathcal{A}$ (see Exercice 2.7). Let $\mu : \mathcal{A}^+ \to [0, \infty]$ be an additive set function. For any $A \subset \Omega$, let

$$\mu^*(A) := \inf\left\{\sum_{n\geq 1} \mu(A_n) : A_n \in \mathcal{A}, A \subset \bigcup_{n\geq 1} A_n\right\},$$

$$\mu^*_+(A) := \inf\left\{\sum_{n\geq 1} \mu(A_n) : A_n \in \mathcal{A}^+, A \subset \bigcup_{n\geq 1} A_n\right\}.$$

Prove that $\mu^*(A) = \mu^*_+(A)$ for every $A \subset \Omega$.

This shows that in the definition of the outer measure of the extension theorem, one may replace the semiring $\mathcal{A}$ with the ring $\mathcal{A}^+$ it generates.

Exercise 4.4 (Approximation theorem)**.** Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $\mathcal{A}_0 \subset \mathcal{F}$ be a ring such that $\sigma(\mathcal{A}_0) = \mathcal{F}$ and μ is σ-finite on $\mathcal{A}_0$. The goal of this exercise is to prove that for any $A \in \mathcal{F}$ with $\mu(A) < \infty$ and for every $\varepsilon > 0$, there exists $A_\varepsilon \in \mathcal{A}_0$ such that $\mu(A \triangle A_\varepsilon) < \varepsilon$. You may proceed as follows :

1. Let us define the outer measure associated to $\mu_{|\mathcal{A}_0}$

   $$\mu^*(A) := \inf\left\{\sum_{n\geq 1} \mu(A_n) : A_n \in \mathcal{A}_0, A \subset \bigcup_{n\geq 1} A_n\right\}, \quad A \subset \Omega.$$

   Justify that μ^* coincide with μ on $\mathcal{F}$.
   Remark : if the ring $\mathcal{A}_0$ is generated by a semi-ring (which will typically be the case, e.g. the Lebesgue measure), then Exercise 4.3 shows that μ^* is the outer-measure constructed in the extension theorem.
   HINT : use the extension and uniqueness theorems.
2. Let $A \in \mathcal{F}$ with $\mu(A) < \infty$ and let $\epsilon > 0$. Justify that there exists a sequence $(A_n)_{n\geq 1} \subset \mathcal{A}_0$ such that $A \subset \bigcup_{n\geq 1} A_n$ and

   $$\sum_{n\geq 1} \mu(A_n) \leq \mu(A) + \epsilon.$$

3. Approximate $\bigcup_{n\geq 1} A_n$ to find $A_\epsilon \in \mathcal{A}_0$ such that $\mu(A \Delta A_\epsilon) < \epsilon$.

Exercise 4.5 (Regularity theorem)**.** Let $(\mathbb{R}^d, \mathcal{R}^d, \mu)$ be a measure space. Assume that μ is finite on compact subsets. Define the following set functions on $\mathcal{R}^d$

$$\mu_1(A) := \sup\{\mu(F) : F \subset A,\ F \text{ compact }\},$$

$$\mu_2(A) := \inf\{\mu(G) : A \subset G,\ G \text{ open }\}.$$

The goal of this exercice is to show that $\mu = \mu_1 = \mu_2$ on $\mathcal{R}^d$.

1. Show that if a Borel measure ν is σ-finite on $\mathcal{A} := \{]a_i, b_i] : -\infty < a_i \leq b_i < \infty\} \subset \mathcal{R}$, we don't always have that $\nu(K) < \infty$ for every compact $K \subset \mathbb{R}$;
   Hint : look at the Dirac measures $\delta_{1/n}(E) := 1_{\{1/n \in E\}}, n \geq 1$ on the Borel subsets of $\mathbb{R}$.
2. Let $A \in \mathcal{R}^d$ be such that $\mu(A) < \infty$. Use the extension and uniqueness theorems as in Exercise 4.4 to prove that for every $\varepsilon > 0$, there exists an open set $G \subset \mathbb{R}^d$ such that $A \subset G$ and $\mu(G) \leq \mu(A) + \epsilon$. Deduce that $\mu_2(A) \leq \mu(A)$ and hence $\mu_2(A) = \mu(A)$. Show that this holds even if $\mu(A) = \infty$.
3. Let $A \in \mathcal{R}^d$ be bounded and let $\epsilon > 0$. There exists a compact $C \subset \mathbb{R}^d$ such that $A \subset C$. Find a suitable open subset $U \subset \mathbb{R}^d$ such that $C \setminus A \subset U$ and $\mu(A) \leq \mu(K) + \epsilon$, where $K := C \setminus U$. Deduce that $\mu(A) \leq \mu_1(A)$ and hence $\mu(A) = \mu_1(A)$. Show that this holds even if $A \in \mathcal{R}^d$ is not bounded.

This proves that any Borel measure on $\mathcal{R}^d$ that is finite on compact sets is *regular*. In particular, this is the case for the Lebesgue measure and any probability measure.

5 Measurable functions

Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces. Let $f : X \to Y$ be a map. We say that f is a $\mathcal{B}, \mathcal{A}$-measurable map if $f^{-1}(B) \in \mathcal{A}$ for every $B \in \mathcal{B}$. If $\mathcal{B} = \sigma(\mathcal{F})$ for some $\mathcal{F} \subset 2^Y$, then f is $\mathcal{B}, \mathcal{A}$-measurable if and only if $f^{-1}(F) \in \mathcal{A}$ for any $F \in \mathcal{F}$.

A map $f : X \to \mathbb{R}^d$ is called measurable, or $\mathcal{A}$-measurable or Borel-measurable, if it is $\mathcal{R}^d$,$\mathcal{A}$-measurable.

A map $f : \mathbb{R}^{d_1} \to \mathbb{R}^{d_2}$ is called Lebesgue-measurable if it is $\mathcal{R}^{d_2}, \mathcal{R}_\lambda^{d_1}$-measurable. Of course, any Borel measurable function is Lebesgue measurable, but the converse is false in general.

Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $f : X \to \mathbb{R}$ be a function. We say that f is a simple function if $f(x) = \sum_{k=0}^n a_k \mathbb{1}_{A_k}(x)$, for some $a_0, \dots, a_n \in \mathbb{R}$, $A_0, \dots, A_n \in \mathcal{A}$ disjoint, and some $n \in \mathbb{N}$. The collection of simple functions is denoted $\mathcal{S}(X, \mathcal{A})$. We also denote $\mathcal{S}^+(X, \mathcal{A})$ the set of non-negative simple functions.

We also adopt the following notation, for any $B \in \mathcal{R}$,

$$\{f \in B\} := \{x \in X : f(x) \in B\}.$$

Theorem. Let $(X, \mathcal{F}, \mu)$ be a measure space and $f : X \to [-\infty, \infty]$ be a measurable map. There exists a sequence (s_n) of simple functions such that $s_n(x) \to f(x)$ for any $x \in X$ as $n \to \infty$.

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Exercise 5.1. Let $(\Omega, \mathcal{F})$ be a measurable space. Prove the following statements.

1. If $f : \mathbb{R}^{d_1} \to \mathbb{R}^{d_2}$ is continuous, then f is measurable ;
2. $f = (f_1, \dots f_k) : \Omega \to \mathbb{R}^k$ is $\mathcal{F}$-measurable if and only if $f_i : \Omega \to \mathbb{R}$ is $\mathcal{F}$-measurable for any $i = 1, \dots, k$;
   HINT : Recall that $\mathcal{R}^k = \sigma(\mathcal{A})$, where $\mathcal{A} := \{\prod_{i=1}^k (a_i, b_i) : a_i, b_i \in \mathbb{R}\}$.
3. If $f_1, \dots, f_d : \Omega \to \mathbb{R}$ are $\mathcal{F}$-measurable and if $g : \mathbb{R}^d \to \mathbb{R}$ is measurable, then $\omega \mapsto g(f_1(\omega), \dots, f_d(\omega))$ is $\mathcal{F}$-measurable.

Exercise 5.2. Let $f : \mathbb{R} \to \mathbb{R}$ be Borel-measurable and let $g : \mathbb{R} \to \mathbb{R}$ be such that $g(x) = f(x)$ for all $x \in \mathbb{R} \setminus N$, where $N \in \mathcal{R}$ is such that $\lambda(N) = 0$.

1. Show that g is Lebesgue-measurable ;
2. If N is at most countable, show that g is Borel-measurable ;
3. If N is uncountable, could it be that g is not Borel-measurable ?

Exercise 5.3. Let $(\Omega, \mathcal{F})$ be a measurable space and let $f_n : \Omega \to [-\infty, \infty]$ be $\mathcal{F}$-measurable functions.

1. If $f_n(\omega)$ converges in $[-\infty, +\infty]$ for all $\omega \in \Omega$, then $\lim_{n\to\infty} f_n$ is $\mathcal{F}$-measurable ;
2. $\{\omega \in \Omega : f_n(\omega) \text{ converges in } [-\infty, \infty], n \to \infty\} \in \mathcal{F}$;
3. If $f : \Omega \to [-\infty, \infty]$ is $\mathcal{F}$-measurable, then $\{\omega \in \Omega : f_n(\omega) \to f(\omega), n \to \infty\} \in \mathcal{F}$.

Exercise 5.4. Let $(X, \mathcal{F})$ be a measurable space. Let $(A_n)_n \subset \mathcal{F}$ be disjoint and $f_n : \Omega \to [-\infty, \infty], n \geq 1$ be measurable functions. Set

$$f(x) := \sum_{k=1}^{\infty} f_k(x) \mathbb{1}_{A_k}(x).$$

Prove that f is well-defined and measurable.

Exercise 5.5. Let $(X, \mathcal{A})$ be a measurable space. Let $f : X \to \mathbb{R}$ be a measurable and bounded function. Show that there exists a sequence of functions $(f_n)_n \subset \mathcal{S}(X, \mathcal{A})$ such that $f_n \to f$ uniformly on X.

HINT : first show the result for $f \geq 0$.

Exercise 5.6 (Egorov's theorem)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $f_j, f : X \to \mathbb{R}$ be measurable functions. Assume that there exists $A \in \mathcal{A}$ such that $\mu(A) < \infty$ and $f_j(x) \overset{j \to \infty}{\to} f(x)$ for all $x \in A$.

1. Let $\varepsilon > 0$ and
$$A_n^m := \bigcap_{j \geq n} \left(\left\{ |f_j - f| < 1/m \right\} \cap A \right).$$
Show that for all $m \geq 0$, there exists $k(m)$ such that $\mu(A \setminus A_{k(m)}^m) \leq \varepsilon 2^{-m}$.
2. Prove that for every $\varepsilon > 0$, there exists $A_\varepsilon \in \mathcal{A}$ such that $A_\varepsilon \subset A$ with $\mu(A \setminus A_\varepsilon) < \varepsilon$ and $f_n \to f$ uniformly on A_ε, i.e.
$$\sup_{x \in A_\varepsilon} |f_n(x) - f(x)| \overset{n \to \infty}{\to} 0.$$

Exercise 5.7 (Lusin's theorem)**.** Let $(\mathbb{R}^d, \mathcal{R}^d, \mu)$ be a measure space. Assume that is μ is finite on compact sets. Let $f : \mathbb{R}^d \to \mathbb{R}$ be measurable. Let $A \in \mathcal{R}^d$ be such that $\mu(A) < \infty$. Prove that for every $\varepsilon > 0$, there exists a compact subset $K_\varepsilon \subset A$ such that $\mu(A \setminus K_\varepsilon) < \varepsilon$ and $f_{|K_\varepsilon} : K_\varepsilon \to \mathbb{R}$ is continuous (for the induced topology on K_ε).

HINT : prove the statement for simple functions by partitioning A in an appropriate way and by using the regularity theorem. In the general case, use Egorov's theorem.

Remark : "$f_{|K}$ is continuous" does not mean that f is continuous on K ! This means that, when considering K as a metric space with metric induced by the usual metric of $\mathbb{R}^d$, $f : K \to \mathbb{R}$ is continuous. It is possible to show that when A is open or closed in $\mathbb{R}^d$, there exists a continuous fonction $f_\varepsilon : \mathbb{R}^d \to \mathbb{R}$ with compact support such that $f = f_\varepsilon$ on K_ε and $\sup_{x \in \mathbb{R}^d} |f_\varepsilon(x)| \leq \sup_{x \in \mathbb{R}^d} |f(x)|$.

Littelwood's three principles of real analysis. The 20th century English mathematician John Littlewood wrote in his 1944 book "Lectures on the theory of functions" (Introduction, section 4 "Theory of functions of a real variable") the following :

> [In the theory of functions of a real variable,] there are three principles, roughly expressible in the following terms : every (measurable) set is nearly a finite union of intervals ; every (measurable) function is nearly continuous ; every convergent sequence of functions is nearly uniformly convergent.

These principles respectively stand for the approximation theorem, Lusin's theorem and Egorov's theorem. The three of them are valid for finite-measure sets.

6 Lebesgue integral I

Let $(X, \mathcal{F}, \mu)$ be a measure space. If $f : X \to [0, \infty]$ is a measurable function, we define

$$\int_X f\,d\mu := \sup\left\{\int_X g\,d\mu : 0 \le g \le f,\ g \in \mathcal{S}^+(X, \mathcal{A})\right\} \in [0, \infty],$$

where $\int_X g\,d\mu = \sum_{k=1}^n a_k\mu(A_k)$ if $g(x) = \sum_{k=1}^n a_k \mathbb{1}_{A_k}(x) \in \mathcal{S}^+(X, \mathcal{A})$, with the convention that $a_k\mu(A_k) = 0$ if $a_k = 0$, even if $\mu(A_k) = \infty$. Note that even if $g < \infty$, we can have $\int_X g\,d\mu = \infty$ if $a_k > 0$ and $\mu(A_k) = \infty$ for some k.

If $f : X \to [-\infty, \infty]$ is measurable, let $f^+ = |f|\mathbb{1}_{\{f\ge 0\}}$ and $f^- = |f|\mathbb{1}_{\{f\le 0\}}$. We define

$$\int_X f\,d\mu := \int_X f^+\,d\mu - \int_X f^-\,d\mu$$

if at least one of the two integrals in the r.h.s. is finite. If both $\int_X f^+\,d\mu$ and $\int_X f^-\,d\mu$ are infinite, the Lebesgue integral of f is undefined. If $\int_X f\,d\mu$ is well-defined, it is sometimes said that f is semi-integrable. If we further have that $\int_X f\,d\mu \in (-\infty, \infty)$ (i.e. $\int_X |f|\,d\mu < \infty$), we say that f is integrable and we write $f \in L^1(X, \mu)$.

For any $A \in \mathcal{F}$, we define

$$\int_A f\,d\mu := \int_X f\mathbb{1}_A\,d\mu.$$

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Exercise 6.1. Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $f, g : X \to \mathbb{R}$ be measurable and semi-integrable functions and $A, B \in \mathcal{A}$. Prove the following statements :

1. if $f \le g$, then $\int_A f\,d\mu \le \int_A g\,d\mu$;
   HINT : show that $f^+ \le g^+$ and $f^- \ge g^-$.
2. if $f \ge 0$ and $A \subset B$, then $\int_A f\,d\mu \le \int_B f d\mu$. Is this still true when f is not non-negative ?
3. if $c \in \mathbb{R} \setminus \{0\}$, then $\int_A cf\,d\mu = c\int_A f\,d\mu$;
4. if $f(x) = 0,\ \forall x \in A$, then $\int_A f\,d\mu = 0$, even if $\mu(A) = \infty$;
5. if $\mu(A) = 0$, then $\int_A f\,d\mu = 0$, even if $f(x) = \infty\ \forall x \in A$;

Exercise 6.2. Let $(\Omega, \mathcal{F}, \mu)$ be a measure-space. Let $f : \Omega \to \mathbb{R}$ be $\mathcal{F}$-measurable and μ-semi-integrable. For every disjoint sequence $(A_n)_n \subset \mathcal{F}$, prove that

$$\int_{\biguplus_{n\ge 1} A_n} f\,d\mu = \sum_{n\ge 1}\int_{A_n} f\,d\mu.$$

Deduce that if $f \ge 0$, then $E \mapsto \int_E f\,d\mu$ is a measure on $\mathcal{A}$.

Exercise 6.3. Let $(X, \mathcal{F}, \mu)$ be a measure space and $f : X \to \mathbb{R}$ be a measurable map.

1. Prove Markov's inequality :

$$\mu(\{|f| > r\}) \le \frac{1}{r}\int_X |f|\,d\mu, \quad \forall\ r > 0\cdot$$

2. Deduce that if $f \geq 0$ and $\int_X f \, d\mu = 0$, then $f = 0$ μ-almost everywhere.

Exercise 6.4 (Monotone convergence, general case)**.** Let $(\Omega, \mathcal{F}, \mu)$ be a measure-space. Let $f_n, f : \Omega \to \mathbb{R}$ be $\mathcal{F}$-measurable functions such that $f_n \nearrow f$ μ-almost everywhere. Prove that if $\int_\Omega f_1^- \, d\mu < \infty$, then the functions f_n and f are μ-semi-integrable and we have $\int_\Omega f_n \, d\mu \nearrow \int_\Omega f \, d\mu$.

HINT : observe that $f_n + f_1^- \geq 0$ for any $n \geq 1$.

Exercise 6.5 (Fatou's lemma, general case)**.** Let $(\Omega, \mathcal{F}, \mu)$ be a measure-space. Let $f_n, f : \Omega \to \mathbb{R}$ be $\mathcal{F}$-measurable functions.

1. If there exists an $\mathcal{F}$-measurable function $g : \Omega \to \mathbb{R}$ such that $f_n \geq g$ for any $n \geq 1$ and $\int_\Omega g^- \, d\mu < \infty$, prove that
$$\int_\Omega \liminf_{n\to\infty} f_n \, d\mu \leq \liminf_{n\to\infty} \int_\Omega f_n \, d\mu.$$
HINT : notice that $\inf_{n\geq 1} f_n \geq g$ and use the monotone convergence theorem.
2. If there exists an $\mathcal{F}$-measurable function $h : \Omega \to \mathbb{R}$ such that $f_n \leq h$ for any $n \geq 1$ and $\int_\Omega h^+ \, d\mu < \infty$, prove that
$$\limsup_{n\to\infty} \int_\Omega f_n \, d\mu \leq \int_\Omega \limsup_{n\to\infty} f_n \, d\mu.$$

Exercise 6.6. Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $f_n, f : X \to \mathbb{R}$ be measurable functions such that $f_n \searrow f$ μ-a.e.. Prove that if $\int_X f_1^+ \, d\mu < \infty$, then the functions f_n and f are μ-semi-integrable and we have
$$\int_X f_n \, d\mu \searrow \int_X f \, d\mu.$$
Give a counterexample when f_1 is not integrable.

Exercise 6.7. Let $(X, \mathcal{A}, \mu)$ be a finite measure space. Let $f_n, f : X \to \mathbb{R}, n \geq 1$ be measurable and such that $f_n \to f$ uniformly on X. Show that if $f_n \in L^1(X, \mu)$ for all $n \geq 1$, then $f \in L^1(X, \mu)$ and we have
$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$
Show that this result is no longer true in general if $\mu(X) = \infty$.

Exercise 6.8. Consider the measure space $(\mathbb{N}, 2^{\mathbb{N}}, \#)$, where $\#$ stands for the counting measure, i.e. $\#A := \#(A \cap \mathbb{N})$. Show that any function $f : \mathbb{N} \to [0, \infty]$ is measurable and that
$$\int_\mathbb{N} f \, d\# = \sum_{n\geq 0} f(n).$$

Exercise 6.9. Let $(X, \mathcal{A}, \delta_a)$ be a measure space, where δ_a is the Dirac measure at some $a \in X$. Let $f : X \to \mathbb{R}$ be measurable. Prove that
$$\int_X f \, d\delta_a = f(a).$$

Exercise 6.10. Let $(X, \mathcal{F})$ be a measurable space. Let (p_n) be a non-negative sequence such that $\sum_{n\in\mathbb{N}} p_n < \infty$ and let $(x_n) \subset X$ be an injective sequence. Consider the measure μ on $\mathcal{F}$ given by $\mu = \sum_{n\in\mathbb{N}} p_n \delta_{x_n}$ (see Exercice 3.9), defined for any $A \in \mathcal{F}$ by
$$\mu(A) = \sum_{n\in\mathbb{N}} p_n \mathbb{1}_A(x_n)\cdot$$
Prove that for any measurable function $f : X \to [0, \infty)$, we have
$$\int_X f \, d\mu = \sum_{n\in\mathbb{N}} p_n f(x_n).$$
Can you caracterize the μ-integrable functions ? What is the value of their integral in this case ?

7 Comparison with the Riemann integral

Let $[a, b] \subset \mathbb{R}$ be a bounded interval. A partition of $[a, b]$ is a finite set

$$P := \{a =: x_0 < x_1 < \ldots < x_n := b\}.$$

The mesh (*le pas*) of the partition P is

$$d(P) := \max_{i=1,\ldots,n}(x_i - x_{i-1}).$$

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Given a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a, b]$, we define a lower sum $L(f, P)$ and an upper sum $U(f, P)$

$$L(f, P) := \sum_{j=1}^{n} m_j(x_j - x_{j-1}) \qquad U(f, P) := \sum_{j=1}^{n} M_j(x_j - x_{j-1}),$$

where $m_j := \inf_{x\in[x_{j-1},x_j]} f(x)$ and $M_j := \sup_{x\in[x_{j-1},x_j]} f(x)$. It is well known that if P_1, P_2 are two partitions of $[a, b]$ such that $P_1 \subset P_2$, then

$$L(f, P_1) \leq L(f, P_2) \leq U(f, P_2) \leq U(f, P_1).$$

Let $\mathcal{P}$ be the collection of partitions of $[a, b]$. Let then

$$L(f) := \sup_{P\in\mathcal{P}} L(f, P) \quad \text{and} \quad U(f) := \inf_{P\in\mathcal{P}} U(f, P).$$

We say that f is Riemann-integrable if $L(f) = U(f)$.

The goal of this session is to study the links between the previous construction, due to Jean-Gaston Darboux (1842-1917) and Bernhard Riemann (1826 - 1866), and the one from Henri Lebesgue (1875 - 1941). A bounded function is not especially Borel-measurable (nor Lebesgue-measurable). For example, if $E \subset [a, b]$ is a non-Lebesgue measurable subset, then $g := \mathbb{1}_E$ is bounded but not a Lebesgue-measurable function. On the contrary, a Borel-measurable function is not always bounded, e.g. $g := 1/x, x \in [0, 1]$.

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Exercise 7.1. Let $f : [a, b] \to \mathbb{R}$ be a bounded function.

1. Justify that there exists a sequence (P_n) of partitions of $[a, b]$ such that $P_n \subset P_{n+1}$ for any n, $d(P_n) \to 0$ and $L(f, P_n) \to L(f)$ as $n \to \infty$. Let (P_n) be such partitions and write, for any $n \geq 1$
$$P_n := \left\{a = x_0^{(n)}, \ldots, x_{k_n}^{(n)} = b\right\}.$$
2. Let $g_n := \sum_{i=1}^{k_n} m_i^{(n)} \mathbb{1}_{]x_{i-1}^{(n)}, x_i^{(n)}[}$, where $m_i^{(n)} := \inf_{x\in[x_{i-1}^{(n)}, x_i^{(n)}]} f(x)$. Justify that g_n is a non-decreasing sequence of functions with $g_n \leq f$. Deduce that there exists a Borel-measurable function g such that $g_n \nearrow g$ pointwise on $[a, b]$ and $g \leq f$. Prove that g is Lebesgue-integrable and that
$$\int_{[a,b]} g \, d\lambda = L(f).$$

3. Using a similar construction, prove that there exists a Borel-measurable function $h : [a,b] \to \mathbb{R}$ such that $f \leq h$, h is Lebesgue-integrable and we have

$$\int_{[a,b]} h \, d\lambda = U(f).$$

4. Conclude that if f is Riemann-integrable, then f is Lebesgue-measurable, Lebesgue-integrable and we have

$$\int_{[a,b]} f \, d\lambda = \int_a^b f(x) \, dx,$$

where $\int_a^b f(x)\, dx$ is the classical Riemann-integral.

Exercise 7.2. Let us use the same notations as in Exercice 7.1. For all $x \in [a,b]$, let $\underline{f}(x) = \lim_{\delta \to 0^+} I_\delta(x)$ and $\overline{f}(x) = \lim_{\delta \to 0^+} S_\delta(x)$, where

$$I_\delta(x) = \inf \Big\{ f(y) \, ; \; y \in [a,b] \cap B(x,\delta) \Big\},$$

$$S_\delta(x) = \sup \Big\{ f(y) \, ; \; y \in [a,b] \cap B(x,\delta) \Big\}.$$

Prove the following statements.

1. For all $x \in [a,b]$, $I_\delta(x)$ is a non-increasing function of δ, while $S_\delta(x)$ is a non-decreasing function of δ.
2. The functions $\underline{f}$ and $\overline{f}$ are well-defined and $\underline{f} \leq f \leq \overline{f}$ on $[a,b]$.
3. Let $x \in [a,b]$. For each $n \geq 1$, there exists a unique $i_n \in \{1, \dots, k_n\}$ such that $x \in [x_{i_n-1}^{(n)}, x_{i_n}^{(n)}]$. Let

$$\delta_n(x) := \min \Big\{ x - x_{i_n-1}^{(n)}, x_{i_n}^{(n)} - x \Big\} \qquad \text{and } \overline{\delta}_n(x) := x_{i_n}^{(n)} - x_{i_n-1}^{(n)}.$$

   Prove that there exists an at most countable set $D \subset [a,b]$ such that for all $x \in [a,b] \setminus D$, we have :

$$I_{\overline{\delta}_n(x)}(x) \leq g_n(x) \leq I_{\delta_n(x)}(x).$$

   Conclude that $\underline{f}(x) = g(x)$ for all $x \in [a,b] \setminus D$, $\underline{f}$ is Borel-measurable, Lebesgue-integrable and

$$\int_{[a,b]} \underline{f} \, d\lambda = L(f).$$

4. Using a similar argument, prove that there exists an at most countable set $D' \subset [a,b]$ such that $\overline{f}(x) = h(x)$ for all $x \in [a,b] \setminus D'$. Justify that $\overline{f}$ is Borel-measurable, Lebesgue-integrable and that we have :

$$\int_{[a,b]} \overline{f} \, d\lambda = U(f).$$

5. Prove that the function f is Riemann-integrable if and only if $\underline{f} = \overline{f}$ almost everywhere on $[a,b]$. Show that $\overline{f}(x) = \underline{f}(x)$ if and only if f is continuous at x. Conclude that f is Riemann-integrable if and only if the set of points of discontinuities of f has Lebesgue measure 0.

Exercise 7.3. Let $f : \mathbb{R} \to \mathbb{R}$ be bounded and Riemann-integrable on $[a,b]$ for every $-\infty < a < b < \infty$.

1. Prove that f is Lebesgue-measurable.
2. Show that if f is semi-integrable (in the Lebesgue sense) on $\mathbb{R}$, then

$$\int_{\mathbb{R}} f \, d\lambda = \lim_{m,M \to \infty} \int_{-m}^{M} f(x) \, dx,$$

   where the last integral is a Riemann integral.

3. Let $f(x) := \sin(x)/x, x \neq 0$, $f(0) := 1$. Show that f is bounded and Riemann-integrable on every interval. Further show that

$$\lim_{m,M\to\infty} \int_{-m}^{M} f(x)\, dx \quad \text{exists in } \mathbb{R},$$

but f is not semi-integrable in the Lebesgue sense.

4. What intuition do you have about the difference between being Riemann-integrable on $\mathbb{R}$ (the limit of the integral on $[-m, M]$ exists) and Lebesgue-integrable on $\mathbb{R}$? Think of the case of continuous functions.

Exercise 7.4. Since $\mathbb{Q}$ is countable, so is $\mathbb{Q} \cap [0, 1]$. We then have $\mathbb{Q} \cap [0, 1] = (q_n)_{n\in\mathbb{N}}$. Let us consider, for all $n \in \mathbb{N}$, the function f_n defined on $[0, 1]$ by

$$f_n(x) = \begin{cases} 1 & \text{if } x \in \{q_0, q_1, \cdots, q_n\}, \\ 0 & \text{otherwise.} \end{cases}$$

1. Show that each f_n is Riemann-integrable and compute

$$\int_0^1 f_n(x)\, dx.$$

2. Show that $f_n \nearrow f := \mathbb{1}_{\mathbb{Q}\cap[0,1]}$ on $[0, 1]$. Is f Riemann-integrable ? Compute $\int_{[0,1]} f\, d\lambda$.

Exercise 7.5. Determine the limits of the following Riemann-integrals

$$\int_0^n \left(1 - \frac{x}{n}\right)^n e^{x/2}\, dx \quad \text{and} \quad \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x}\, dx.$$

HINT : use the fact that $n \mapsto (1 + x/n)^n$ is increasing if $x > -n$.

Exercise 7.6. Consider the functions

$$f_1(x) := \begin{cases} +\infty & \text{if } x = 0 \\ \ln|x| & \text{if } 0 < |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}, \quad f_2(x) := \begin{cases} \frac{1}{x^2-1} & \text{if } |x| \neq 1 \\ 42 & \text{if } |x| = 1 \end{cases}, \quad f_3(x) = 1.$$

Determine whether each of these functions is integrable on $(\mathbb{R}, \mathcal{R})$ with respect to the measure m in the following cases and compute, if possible, the value of the integrals

1. $m = \lambda$ is the Lebesgue measure ;
2. m is the measure defined by

$$m(B) := \sum_{n\in B\cap\mathbb{Z}} \frac{1}{1 + (n + 1)^2},$$

for each $B \in \mathcal{R}$.
HINT : the primitive of $\frac{1}{x^2-1}$ is $\frac{1}{2} \log |\frac{x-1}{x+1}|$.

8 Lebesgue integral II

Exercise 8.1. Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $f_n, f : X \to \mathbb{R} \in L^1(X, \mu)$ and $\varphi_n, \varphi : X \to \mathbb{R}$ be measurable functions. Assume that there exists $B > 0$ such that $|\varphi_n(x)| \leq B$ for any $x \in X$ and any $n \geq 1$. Further assume that $\int_X |f_n - f| \, d\mu \to 0$ and that $\varphi_n \to \varphi$ μ-a.e. Prove that $\varphi_n f_n \in L^1(X, \mu)$, $\varphi f \in L^1(X, \mu)$ and that $\int_X |\varphi_n f_n - \varphi f| \, d\mu \to 0$.

Exercise 8.2 (Continuity of the integral)**.** Let $(X, \mathcal{F}, \mu)$ be a measure space. Let $f : X \to [-\infty, \infty]$ be such that $f \in L^1(X, \mu)$. Prove that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $A \in \mathcal{F}$ with $\mu(A) < \delta$, we have $\int_A |f| \, d\mu < \varepsilon$. Give an example of a function $f : X \to [-\infty, \infty]$ with the last property such that $f \notin L^1(X, \mu)$.

Hint : first prove the statement for $f \geq 0$ by showing that for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\int_{\{f > N\}} f \, d\mu < \varepsilon$.

Exercise 8.3. Let $(\mathbb{R}^d, \mathcal{R}^d, \mu)$ be a measure space such that μ is finite on compact sets. Let $f : \mathbb{R}^d \to \mathbb{R} \in L^1(\mathbb{R}, \mathcal{R}, \mu)$ and assume that $\int_K f \, d\mu = 0$ for any compact subset $K \subset \mathbb{R}^d$. Show that $f = 0$ μ-almost everywhere.

Hint : Use the regularity theorem and the continuity property of the integral of Exercice 8.2.

Exercise 8.4. Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $f : X \to \mathbb{R}$ be such that $f \in L^1(X, \mu)$ and $f(x) \neq 0$ for μ-almost all $x \in X$, i.e. $\mu(\{f = 0\}) = 0$.

1. Show that if $\mu(\{|f| < c_0\}) < \infty$ for some $c_0 > 0$, then for every $\varepsilon > 0$ there exists $\delta_\epsilon > 0$ such that for every $A \in \mathcal{A}$ with $\mu(A) \geq \varepsilon$, we have $\int_A |f(x)| \, d\mu \geq \delta_\epsilon$. In particular, this result is always true when $\mu(X) < \infty$.

   Hint : justify that $A \subset \{|f| < c\} \uplus \Big(A \cap \{|f| \geq c\}\Big)$ and estimate $\int_A |f| \, d\mu$ by $\int_{A \cap \{|f| \geq c\}} |f| \, d\mu$.
2. Give an example to show that the previous result is not valid in general if $\mu(X) = \infty$.

Exercise 8.5. Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and let $f : \Omega \to \mathbb{R}$ be a measurable function. Let $C \subset \mathbb{R}$ be a closed set and assume that

$$\frac{1}{\mu(A)} \int_A f \, d\mu \in C,$$

for any $A \in \mathcal{R}$ with $\mu(A) > 0$. Prove that $f(\omega) \in C$ for μ-almost every $\omega \in \Omega$.

9 Product spaces

Let $(\Omega_1, \mathcal{F}_1, \mu_1), (\Omega_2, \mathcal{F}_2, \mu_2)$ be measure spaces. We define $\mathcal{F}_1 \times \mathcal{F}_2 := \{A_1 \times A_2 : A_i \in \mathcal{F}_i\}$ and

$$\mathcal{F}_1 \otimes \mathcal{F}_2 := \sigma(\mathcal{F}_1 \times \mathcal{F}_2) = \sigma\Big(\{A_1 \times A_2 : A_i \in \mathcal{F}_i\}\Big).$$

Theorem. If μ_1 and μ_2 are σ-finite, then there exists a unique measure on $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$, denoted $\mu_1 \otimes \mu_2$, such that $(\mu_1 \otimes \mu_2)(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2)$ for all $A_i \in \mathcal{F}_i$. This measure satisfies

$$\begin{aligned}(\mu_1 \otimes \mu_2)(A) &= \int_{\Omega_1} \Big(\int_{\Omega_2} \mathbb{1}_A(\omega_1, \omega_2)\, d\mu_2(\omega_2) \Big)\, d\mu_1(\omega_1) \\ &= \int_{\Omega_2} \Big(\int_{\Omega_1} \mathbb{1}_A(\omega_1, \omega_2)\, d\mu_1(\omega_1) \Big)\, d\mu_2(\omega_2),\end{aligned}$$

for all $A \in \mathcal{F}_1 \otimes \mathcal{F}_2$.

Proposition. Let $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be $\mathcal{F}_1 \otimes \mathcal{F}_2$-measurable. Then $\omega_2 \mapsto f(\omega_1, \omega_2) =: f_{\omega_1}(\omega_2)$ is $\mathcal{F}_2$-measurable for all $\omega_1 \in \Omega_1$ and $\omega_1 \mapsto f(\omega_1, \omega_2) =: f_{\omega_2}(\omega_1)$ is $\mathcal{F}_1$-measurable for all $\omega_2 \in \Omega_2$.

Theorem. Let $f : \Omega_1 \times \Omega_2 \to [0, \infty]$ be an $\mathcal{F}_1 \otimes \mathcal{F}_2$-measurable non-negative map. If μ_1 and μ_2 are σ-finite, then $\omega_1 \mapsto \int_{\Omega_2} f_{\omega_1}\, d\mu_2$ is $\mathcal{F}_1$-measurable, $\omega_2 \mapsto \int_{\Omega_1} f_{\omega_2}\, d\mu_1$ is $\mathcal{F}_2$-measurable, and we have that

$$\int_{\Omega_1 \times \Omega_2} f\, d(\mu_1 \otimes \mu_2) = \int_{\Omega_1} \Big(\int_{\Omega_2} f\, d\mu_2 \Big)\, d\mu_1 = \int_{\Omega_2} \Big(\int_{\Omega_1} f\, d\mu_1 \Big)\, d\mu_2.$$

Theorem. Let $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be $\mathcal{F}_1 \otimes \mathcal{F}_2$-measurable. Assume that μ_1 and μ_2 are σ-finite and that $\int_{\Omega_1 \times \Omega_2} |f|\, d(\mu_1 \otimes \mu_2) < \infty$. Then, f_{ω_1} is μ_2-integrable for μ_1-almost every $\omega_1 \in \Omega_1$, f_{ω_2} is μ_1-integrable for μ_2-almost every $\omega_2 \in \Omega_2$ and

$$\int_{\Omega_1 \times \Omega_2} f\, d(\mu_1 \otimes \mu_2) = \int_{\Omega_1} \Big(\int_{\Omega_2} f\, d\mu_2 \Big)\, d\mu_1 = \int_{\Omega_2} \Big(\int_{\Omega_1} f\, d\mu_1 \Big)\, d\mu_2.$$

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Exercise 9.1. Prove that $\mathcal{R}^k \otimes \mathcal{R}^m = \mathcal{R}^{k+m}$.

HINT : Consider the projections $\pi_j : \mathbb{R}^{k+m} \to \mathbb{R}^j$ to prove $\subset$.

Exercise 9.2. Let $(x_{n,k})_{n,k \geq 1} \subset [0, \infty)$. Prove by hand (i.e. without using Fubini's theorem), that

$$\sum_{k \geq 1} \sum_{n \geq 1} x_{n,k} = \sum_{n \geq 1} \sum_{k \geq 1} x_{n,k}$$

Exercise 9.3. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $f_n : \Omega \to \mathbb{R}$ be measurable functions. Assume that μ is σ-finite and that $\sum_{n \geq 1} \int_\Omega |f_n|\, d\mu < \infty$. Prove that

$$\sum_{n \geq 1} \int_\Omega f_n\, d\mu = \int_\Omega \sum_{n \geq 1} f_n\, d\mu.$$

Exercise 9.4. Consider the measure space $(\mathbb{R}^2, \mathcal{R}^2, \lambda \otimes \#)$. Compare the integrals

$$\int_{\mathbb{R}} \Big(\int_{\{a\}} d\# \Big) \, d\lambda \quad \text{and} \quad \int_{\mathbb{R}} \Big(\int_{\{a\}} d\lambda \Big) \, d\#$$

$$\int_{[0,1]} \Big(\int_{[0,1]} \mathbb{1}_D \, d\lambda \Big) \, d\# \quad \text{and} \quad \int_{[0,1]} \Big(\int_{[0,1]} \mathbb{1}_D \, d\# \Big) \, d\lambda,$$

where $D := \{(x,x) : x \in \mathbb{R}\} \subset \mathbb{R}^2$. Is there any contradiction with Fubini's theorem ?

Exercise 9.5. Let $f(x,y) := e^{-xy} - 2e^{-2xy}$. Show that

$$\int_0^1 \Big(\int_1^\infty f(x,y) \, dy \Big) \, dx \neq \int_1^\infty \Big(\int_0^1 f(x,y) \, dx \Big) \, dy.$$

Justify, at each step that the operations you apply on the integrals are legitimate.

Exercise 9.6 (Layer-cake representation)**.** Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space. Let $f : \Omega \to [0,\infty]$ be a measurable function and let

$$A(f) := \Big\{ (\omega, y) : 0 < y < f(\omega) \Big\} \quad \text{and} \quad \tilde{A}(f) := \Big\{ (\omega, y) : 0 < y \leq f(\omega) \Big\}.$$

1. Show that $A(f) \in \mathcal{F} \otimes \mathcal{R}_{[0,\infty)}$ and $\tilde{A}(f) \in \mathcal{F} \otimes \mathcal{R}_{[0,\infty)}$.
2. Show that we have for all $\omega \in \Omega$

   $$f(\omega) = \int_0^\infty \mathbb{1}_{\{f > y\}}(\omega) \, dy = \int_0^\infty \mathbb{1}_{\{f \geq y\}}(\omega) \, dy.$$

   Check that $\mathbb{1}_{A(f)}(\omega, y) = \mathbb{1}_{(y,\infty)}(f(\omega))$ and $\mathbb{1}_{\tilde{A}(f)}(\omega, y) = \mathbb{1}_{[y,\infty)}(f(\omega))$ for all $(\omega, y) \in \Omega \times [0,\infty)$. Show that,

   $$\int_\Omega f \, d\mu = (\mu \otimes \lambda)(A(f)) = (\mu \otimes \lambda)(\tilde{A}(f)).$$

3. Deduce that $G(f) \in \mathcal{F} \otimes \mathcal{R}_{[0,\infty)}$ and that $(\mu \otimes \lambda)(G(f)) = 0$, where

   $$G(f) := \Big\{ (\omega, f(\omega)) : \omega \in \Omega \Big\},$$

   is the graph of f.
4. Justify that the functions $t \mapsto \mu(\{f > t\})$ and $t \mapsto \mu(\{f \geq t\})$ are $\mathcal{R}_{[0,\infty)}$-measurable and prove that

   $$\int_\Omega f \, d\mu = \int_0^\infty \mu(\{f > t\}) \, dt = \int_0^\infty \mu(\{f \geq t\}) \, dt.$$

5. Prove that the set

   $$D := \Big\{ t \in [0,\infty) : \mu(\{f = t\}) > 0 \Big\}$$

   is at most countable. In particular, $\mu(\{f > t\}) = \mu(\{f \geq t\})$ for all $t \geq 0$ except at most countably many of them, hence also for λ-almost every $t \geq 0$. To do so, let $\Omega = \bigcup_{n \geq 1} \Omega_n, \Omega_n \in \mathcal{F}, \mu(\Omega_n) < \infty, n \geq 1$. Justify that we can assume that the collection $(\Omega_n)_{n \geq 1}$ is disjoint. Let

   $$D_{n,k} := \Big\{ t \in [0,\infty) : \mu(\{f = t\} \cap \Omega_n) > \frac{1}{k} \Big\},$$

   for every $n, k \geq 1$. Justify that $\bigcup_{n \geq 1} \bigcup_{k \geq 1} D_{n,k} = D$ and prove that $D_{n,k}$ is finite for every $n, k \geq 1$. Conclude.

Exercise 9.7 (Hyper-spherical coordinates)**.** Let $S^{d-1} := \{x \in \mathbb{R}^d : |x| = 1\}$ be the unit sphere of $\mathbb{R}^d$. Let $\varphi : \mathbb{R}^d \setminus \{0\} \to (0,\infty) \times S^{d-1}$, $\varphi(x) := (|x|, \frac{x}{|x|})$.

1. Justify that φ is a homeomorphism. Deduce that φ is $(0,\infty) \otimes S^{d-1} - \mathcal{R}^d_{\mathbb{R}^d \setminus \{0\}}$-measurable and that φ^{-1} is $\mathcal{R}^d_{\mathbb{R}^d \setminus \{0\}} - (0,\infty) \otimes S^{d-1}$-measurable.
2. For any Borel subset $A \subset S^{d-1}$, let
$$\sigma(A) := d\, \lambda\Big(\varphi^{-1}\Big((0,1) \times A\Big)\Big).$$
Prove that σ is a finite measure on the Borel subsets of $\mathcal{S}^{d-1}$. It is called the *surface area measure.*
3. Compute $\sigma(S^0)$, $\sigma(S^1)$ and $\sigma(S^2)$. Does-it make sense ?
4. Let $f : \mathbb{R}^d \to [-\infty, \infty]$ be a Borel-measurable function. Justify that the map
$$(0,\infty) \times S^{d-1} \to \mathbb{R}, (r,u) \mapsto f(ru)$$
is $\mathcal{R} - (0,\infty) \otimes S^{d-1}$-measurable. Deduce that for every $B \in \mathcal{R}^d$, the map
$$(0,\infty) \times S^{d-1} \to \mathbb{R}, (r,u) \mapsto \mathbb{1}_B(ru)$$
is $\mathcal{R} - (0,\infty) \otimes S^{d-1}$-measurable.
5. Let us define the set function μ on $\mathcal{R}^d$ as follows :
$$\mu(B) := \int_0^\infty r^{d-1} \Big(\int_{S^{d-1}} \mathbb{1}_B(ru)\, d\sigma(u) \Big)\, dr,$$
for any $B \in \mathcal{R}^d$. Justify that μ is well defined and prove that μ is a measure on $\mathcal{R}^d$.
6. (a) Prove that
$$\lambda\Big(\varphi^{-1}\Big((0,b] \times A\Big)\Big) = b^d\, \frac{\sigma(A)}{d},$$
for any $b > 0$ and every Borel subset $A \subset S^{d-1}$.
   (b) Deduce that
$$\lambda\Big(\varphi^{-1}\Big((a,b] \times A\Big)\Big) = \mu\Big(\varphi^{-1}\Big((a,b] \times A\Big)\Big),$$
for every $0 \le a < b < \infty$ and every $A \in \mathcal{R}^d_{S^{d-1}}$.
7. (a) Let $A \in \mathcal{R}^d_{\mathcal{S}^{d-1}}$. Let us define
$$\nu_1(B) := \lambda(\varphi^{-1}(B \times A)) \quad \text{and} \quad \nu_2(B) := \mu(\varphi^{-1}(B \times A)),$$
for any $B \in \mathcal{R}_{(0,\infty)}$. Show that ν_1 and ν_2 are measures on $\mathcal{R}_{(0,\infty)}$ and prove that $\nu_1(B) = \nu_2(B)$ for any $B \in \mathcal{R}_{(0,\infty)}$.
   (b) Deduce that $\lambda(\varphi^{-1}(E)) = \mu(\varphi^{-1}(E))$ for all $E \in (0,\infty) \otimes S^{d-1}$.
   (c) Conclude that $\lambda(B) = \mu(B)$ for any $B \in \mathcal{R}^d_{\mathbb{R}^d \setminus \{0\}}$. Deduce that $\lambda = \mu$ on $\mathcal{R}^d$.
8. Prove that for any $f \in \mathcal{S}^+(\mathbb{R}^d, \mathcal{R}^d)$, we have
$$\int_{\mathbb{R}^d} f(x)\, dx = \int_0^\infty r^{d-1} \Big(\int_{S^{d-1}} f(ru)\, d\sigma(u) \Big)\, dr.$$
Conclude that this still holds for any Borel-measurable function $f : \mathbb{R}^d \to [0,\infty]$.
9. (a) Show that $f(x) := (1+|x|^2)^{-m/2} \in L^1(\mathbb{R}^d)$ if and only if $m > d$.
   Hint : show that $(2r^m)^{-1} \le (1+r^2)^{-m/2} \le r^{-m}$ for all $r \ge 1$.
   (b) Show that $g(x) := (1+|x|)^{-m} \in L^1(\mathbb{R}^d)$ if and only if $m > d$.
   Hint : show that $2^{-1}(1+r)^{-m} \le (2r^m)^{-1}$ and $r^{-m} \le 2(1+r)^{-m}$ for all $r \ge 1$.

10 Types of convergence and L^p-spaces

Let $(X, \mathcal{A}, \mu)$ be a measure space and $f : X \to \mathbb{R}$ be a measurable function. Let us recall that for all $p \in [1, \infty)$

$$\|f\|_{L^p(X,\mathcal{A},\mu)} := \left(\int_X |f|^p \, d\mu \right)^{1/p}.$$

When $p = \infty$,

$$\|f\|_{L^\infty(X,\mathcal{A},\mu)} := \inf_{A \in \mathcal{N}} \sup_{x \in X \setminus A} |f(x)|,$$

where $\mathcal{N} := \{A \in \mathcal{A} : \mu(A) = 0\}$. Let us recall that the so-called L^p norm is actually not a norm on the space $\mathcal{L}^p(X, \mathcal{A}, \mu) := \{f : X \to \mathbb{R} : f \text{ measurable and } \|f\|_{L^p(X,\mathcal{A},\mu)} < \infty\}$. It only is a semi-norm as $\|f\|_{L^p} = 0$ does not imply $f = 0$, but only $f = 0$ μ-a.e. However, it is a norm on the vector space $L^p(X, \mathcal{A}, \mu) := \mathcal{L}^p(X, \mathcal{A}, \mu)/\mathcal{N}$ of equivalence classes of $\mathcal{L}^p$ functions that are equal μ-a.e.

For every $p \in [1, \infty]$, $L^p(X, \mathcal{A}, \mu)$ is a Banach space and $\mathcal{S}(X, \mathcal{A}) \cap L^p(X, \mathcal{A}, \mu)$ is dense in $L^p(X, \mathcal{A}, \mu)$ for the $L^p(X, \mathcal{A}, \mu)$-norm.

The L^∞-norm is also called "essential supremum" and denoted supess(f). Similarly, one can define the essential infimum infess$(f) := \sup_{A \in \mathcal{N}} \inf_{x \in X \setminus A} |f(x)|$.

Theorem (Hölder's inequality)**.** Let $p \in [1, \infty]$ and $p' := p/(p-1) \in [1, \infty]$. Let $f, g : X \to \mathbb{R}$ be measurable maps such that $f \in L^p(\mu)$ and $g \in L^{p'}(\mu)$. Then $fg \in L^1(\mu)$ and

$$\|fg\|_{L^1(X,\mu)} = \int_X |fg| \, d\mu \leq \left(\int_X |f|^p \, d\mu \right)^{1/p} \left(\int_X |g|^{p'} \, d\mu \right)^{1/p'} = \|f\|_{L^p(X,\mu)} \|g\|_{L^{p'}(X,\mu)}.$$

Definition. Let $f, f_n : X \to \mathbb{R}$ be measurable maps. We define the following modes of convergence :

1. $f_n \to f$ μ-a.e. if there exists $E \in \mathcal{A}$ such that $\mu(X \setminus E) = 0$ and $f_n(x) \to f(x)$ for any $x \in E$;
2. $f_n \to f$ in μ-measure if $\lim_{n\to\infty} \mu(\{|f_n - f| > \varepsilon\}) = 0$ for any $\varepsilon > 0$;
3. $f_n \to f$ in $L^p(X, \mu)$, $p \in [1, \infty]$, if $\|f_n - f\|_{L^p(X,\mu)} \to 0$;
4. $f_n \to f$ almost uniformly if for every $\varepsilon > 0$, there exists $X_\varepsilon \in \mathcal{A}$ such that $\mu(X \setminus X_\varepsilon) < \varepsilon$ and $f_n \to f$ uniformly on X_ε.

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* *

Exercise 10.1. Let us define the following sequences of functions on $\mathbb{R}$:

$$f_n(x) = \mathbb{1}_{[n,n+1]}(x),$$

$$g_n(x) = n\mathbb{1}_{[0,1/n]}(x),$$

$$h_n(x) = 2^k \mathbb{1}_{[j2^{-k},(j+1)2^{-k}]}, \quad \text{if } n = 2^k + j,$$

with $k \geq 1$ and $j \in \{0, \ldots, 2^k - 1\}$. Considering the Lebesgue measure, determine for each map whether convergence almost-everywhere, in measure and in $L^p(\mathbb{R})$ hold.

Exercise 10.2. Let $(X, \mathcal{F}, \mu)$ be a measure space and let $f, f_n : X \to \mathbb{R}, n \geq 1$ be measurable maps. Prove that the following statements are equivalent.

(i) $f_n \to f$ μ-a.e ;

(ii) $\mu(\limsup_{n\geq 1}\{|f_n - f| > \varepsilon\}) = 0$, for every $\varepsilon > 0$.

HINT : show that for $\varepsilon > 0$ we have

$$\limsup_{n\geq 1}\{|f_n - f| > 2\varepsilon\} \subset \{\limsup_{n\to\infty} |f_n - f| > \varepsilon\} \subset \limsup_{n\geq 1}\{|f_n - f| > \varepsilon\}.$$

Exercise 10.3. Let $(X, \mathcal{F}, \mu)$ be a measure space and let $f, f_n : X \to \mathbb{R}, n \geq 1$ be measurable maps. Prove the following statements.

1. If $\lim_{n\to\infty} \mu(\cup_{k=n}^{\infty}\{|f_k - f| > \varepsilon\}) = 0$ for every $\varepsilon > 0$, then $f_n \to f$ μ-a.e. as $n \to \infty$.
2. If $\mu(X) < \infty$ and if $f_n \to f$ μ-a.e., then for every $\varepsilon > 0$, we have
$$\lim_{n\to\infty} \mu(\cup_{k=n}^{\infty}\{|f_k - f| > \varepsilon\}) = 0.$$
3. If $\mu(X) < \infty$ and if $f_n \to f$ μ-a.e., then $f_n \to f$ in μ-measure.

Exercise 10.4. Let $(X, \mathcal{F}, \mu)$ be a measure space and let $f, f_n : X \to \mathbb{R}, n \geq 1$ be measurable maps. For any $\varepsilon > 0$, assume that there exists $n_\varepsilon \geq 1$ such that $\sum_{n=n_\varepsilon}^{\infty} \mu(\{|f_n - f| > \varepsilon\}) < \infty$. Prove that $f_n \to f$ μ-a.e..

Exercise 10.5. Let $(X, \mathcal{F}, \mu)$ be a measure space and let $f, f_n : X \to \mathbb{R}, n \geq 1$ be measurable maps. Show that if $f_n \to f$ in μ-measure as $n \to \infty$, then any subsequence $(f_{n_k})_{k\geq 1}$ of $(f_n)_{n\geq 1}$ admits further subsequence $(f_{n_{k_j}})_{j\geq 1}$ such that $f_{n_{k_j}} \to f$ μ-a.e. as $j \to \infty$. Show that the converse is true if $\mu(X) < \infty$.

HINT : use exercise 10.4 to prove $\Rightarrow$: show that any subsequence $(f_{n_k})_{k\geq 1}$ admits a further subsequence $(f_{n_{k_j}})_{j\geq 1}$ such that $\mu(\{|f_{n_{k_j}} - f| > 2^{-j}\}) < 2^{-j}$.

When $\mu(X) < \infty$, this is obivously an equivalence. This result contrasts with the convergence of numerical sequences in $\mathbb{R}$! This also proves that there is no metric on the space of measurable functions associated to almost everywhere convergence.

Exercise 10.6. Let $(X, \mathcal{F}, \mu)$ be a measure space. Let $f, f_n : X \to \mathbb{R}$, $n \geq 1$, be measurable functions. Assume that $f_n \to f$ in μ-measure. Show that monotone's convergence and Lebesgue's dominated convergence theorems hold, i.e. prove the following :

1. If the sequence $(f_n)_{n\geq 1}$ is monotone non-decreasing and if $\int_X f_1^- \, d\mu < \infty$, then f is μ-semi-integrable and $\int_X f_n \, d\mu \nearrow \int_X f \, d\mu$.
2. If there exists $g \in L^1(X, \mu)$ such that for all $n \geq 1$ we have $|f_n| \leq g$ μ-a.e., then $f \in L^1(X, \mu)$ and $f_n \to f$ in $L^1(X, \mu)$. In particular $\int_X f_n \, d\mu \to \int_X f \, d\mu$.

Exercise 10.7. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $X, X_n : \Omega \to \mathbb{R}$, $n \geq 1$ be random variables (i.e. measurable functions) and assume that $X_n \to X$ in probability (i.e. in $\mathbb{P}$-measure). Prove that $\varphi(X_n) \to \varphi(X)$ in probability for all $\varphi \in \mathcal{C}^0(\mathbb{R})$. Also prove that $\psi(X_n) \to \psi(X)$ in $L^1(\Omega, \mathbb{P})$ for all $\psi \in \mathcal{C}^0(\mathbb{R}) \cap L^\infty(\mathbb{R})$, hence in particular $\mathbb{E}[\psi(X_n)] \to \mathbb{E}[\psi(X)]$, i.e.

$$\int_\Omega \psi(X_n(\omega)) \, d\mathbb{P}(\omega) \to \int_\Omega \psi(X(\omega)) \, d\mathbb{P}(\omega) \cdot$$

HINT : use Exercise 10.5.

Exercise 10.8. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $X, X_n : \Omega \to \mathbb{R}$, $n \geq 1$ be random variables. Recall that for every $a, b \in \mathbb{R}$ we write $a \wedge b := \min(a, b)$ and $a \vee b := \max(a, b)$. Prove that $X_n \to X$ in probability if and only if $\mathbb{E}[|X_n - X| \wedge 1] \to 0$.

HINT : use Exercise 10.7 to prove $\Rightarrow$, and show that

$$\mathbb{P}(|X_n - X| > \varepsilon) \leq (1 + \varepsilon^{-1})\mathbb{E}[|X_n - X| \wedge 1],$$

for every $\varepsilon > 0$ to prove $\Leftarrow$.

Exercise 10.9. Let $(X, \mathcal{F}, \mu)$ a measure space with $\mu(X) > 0$.

1. Let $1 \leq p \leq q \leq \infty$. Show that if $\mu(X) < \infty$, we have for all $u : X \to \mathbb{R} \in L^q(X, \mu)$ that

$$\frac{\|u\|_{L^p(X,\mu)}}{\mu(X)^{1/p}} \leq \frac{\|u\|_{L^q(X,\mu)}}{\mu(X)^{1/q}},$$

Deduce that if $\mu(X) < \infty$ we have $L^q(X, \mu) \subset L^p(X, \mu)$ for all $1 \leq p < q \leq \infty$. In particular, for all $p \in [1, \infty]$ we have

$$\bigcap_{r\in[p,\infty]} L^r(X, \mu) = L^\infty(X, \mu).$$

Hint : use Hölder's inequality.

2. Let $1 \leq p < q < \infty$. Let $u, v : \mathbb{R} \to \mathbb{R}$ be defined by

$$u(x) = \frac{1}{x^{1/q}} 1_{(0,1]}(x) \quad \text{and} \quad v(x) = \frac{1}{x^{1/p}} 1_{[1,\infty)}(x).$$

Consider the Lebesgue measure on $\mathbb{R}$. Show that *i)* $u \in L^p(\mathbb{R})$ but $u \notin L^q(\mathbb{R})$ and *ii)* $v \in L^q(\mathbb{R})$ but $v \notin L^p(\mathbb{R})$. Do we have $L^p(\mathbb{R}) \subset L^\infty(\mathbb{R})$ or $L^\infty(\mathbb{R}) \subset L^p(\mathbb{R})$?

3. Let $1 \leq p \leq q \leq \infty$ and $u : X \to \mathbb{R}$ be measurable. Let $\theta \in [0, 1]$. Show that

$$\|u\|^{\theta p+(1-\theta)q}_{L^{\theta p+(1-\theta)q}(X,\mu)} \leq \|u\|^{\theta p}_{L^p(X,\mu)} \|u\|^{(1-\theta)q}_{L^q(X,\mu)}$$

Deduce that $\phi(p) := \|u\|^p_{L^p(X,\mu)}$ is a log-convex function (i.e. $\log(\phi)$ is a convex function) on $(1, \infty)$ and that

$$\bigcap_{r\in[p,q]} L^r(X, \mu) = L^p(X, \mu) \cap L^q(X, \mu).$$

Show that $\|u\|^q_{L^q(X,\mu)} \leq \|u\|^{q-p}_{L^\infty(X,\mu)} \|u\|^p_{L^p(X,\mu)}$. Conclude that for all $p \in [1, \infty]$,

$$\bigcap_{r\in[p,\infty]} L^r(X, \mu) = L^p(X, \mu) \cap L^\infty(X, \mu).$$

4. Consider the Lebesgue measure on $\mathbb{R}$.
   (a) Construct a non-negative continuous function (even $\mathcal{C}^\infty$ if you want) $f : \mathbb{R} \mapsto [0, \infty)$ such that $f \in L^1(\mathbb{R})$ but such that we don't have $\lim_{x\to+\infty} f(x) = 0$.
   Hint : consider triangles with base of length $1/k^3$ and height $2k$, $k \geq 1$.
   (b) Construct a non-negative continuous function $f : \mathbb{R} \mapsto [0, \infty)$ such that $f \in \bigcap_{p\in[1,\infty)} L^p(\mathbb{R})$ but $f \notin L^\infty(\mathbb{R})$. What if $\mathbb{R}$ is replaced by a compact or a bounded open subset of $\mathbb{R}$?
   (c) Construct a non-negative continuous function $f : \mathbb{R} \mapsto [0, \infty)$ such that $f \in \bigcap_{p\in[1,\infty]} L^p(\mathbb{R})$ but such that we don't have $\lim_{x\to+\infty} f(x) = 0$.

5. Let $1 \leq p < \infty$. Let us define

$$\ell^p(\mathbb{N}) := L^p(\mathbb{N}, 2^{\mathbb{N}}, \#) = \left\{ (x_k)_{k\geq 0} : \|(x_k)\|_{\ell^p(\mathbb{N})} := \Big(\sum_{k\geq 0} |x_k|^p \Big)^{1/p} < \infty \right\}.$$

The space $\ell^\infty(\mathbb{N}) := L^\infty(\mathbb{N}, 2^{\mathbb{N}}, \#)$ is the collection of bounded sequences, with $\|(x_k)\|_{\ell^\infty(\mathbb{N})} = \sup_{k\geq 0} |x_k|$.
   (a) Show that for all $r \geq 1$ and all $x \geq 0$, we have $(1 + x)^r \geq 1 + x^r$. Deduce that $(a + b)^r \geq a^r + b^r$ for all $a, b \geq 0$ and $1 \leq r < \infty$.
   (b) Let $(x_k)_{k\geq 0}$. Show that for all $n \geq 0$ and all $1 \leq r < \infty$ we have

$$\Big(\sum_{k=0}^{n} |x_k|^r \Big)^{1/r} \leq \sum_{k=0}^{n} |x_k|.$$

(c) Let $(x_k)_{k\geq 0}$. Prove that for all $n \geq 0$ and all $1 \leq p \leq q < \infty$ we have

$$\Big(\sum_{k=0}^{n} |x_k|^q\Big)^{1/q} \leq \Big(\sum_{k=0}^{n} |x_k|^p\Big)^{1/p} \leq \frac{(n+1)^{1/p}}{(n+1)^{1/q}}\Big(\sum_{k=0}^{n} |x_k|^q\Big)^{1/q}.$$

HINT : to prove the second inequality, use Hölder's inequality on the probability space $(\mathbb{N}, 2^{\mathbb{N}}, \nu_n)$ where $\nu_n(A) := (n+1)^{-1}\#(A \cap \{0, 1, \ldots, n\})$, $A \subset \mathbb{N}$ is the uniform probability measure on $\{0, \ldots, n\}$.

In particular, this proves that, for a fixed $n \geq 1$, the p-norm and the q-norm on $\mathbb{R}^n$ are equivalent for all $p, q \in [1, \infty)$. There is a more general result : in any finite-dimensional normed vector space all norms are equivalent.

(d) Let $(x_k)_{k\geq 0}$. Show that $\|(x_k)\|_{\ell^q(\mathbb{N})} \leq \|(x_k)\|_{\ell^p(\mathbb{N})}$ for all $1 \leq p \leq q \leq \infty$ and hence that $\ell^p(\mathbb{N}) \subset \ell^q(\mathbb{N})$. Show that $\ell^q(\mathbb{N}) \subset \ell^p(\mathbb{N})$ is false in general.

This contrasts strongly with the finite-measure case : the reverse inclusion actually holds!

6. Prove that for all $1 \leq p \leq \infty$ we have

$$\bigcap_{r\in[p,\infty]} \ell^r(\mathbb{N}) = \ell^p(\mathbb{N}).$$

Exercise 10.10 (Scheffé's lemma)**.** Let $(X, \mathcal{F}, \mu)$ be a measure space and let $p \in [1, \infty)$. Let $f, f_n : X \to \mathbb{R} \in L^p(\mu)$, $n \geq 1$.

1. Assume that $f_n \to f$ μ-a.e. Prove that if $\|f_n\|_{L^p(\mu)} \to \|f\|_{L^p(\mu)}$, then $\|f_n - f\|_{L^p(\mu)} \to 0$. Show that this is also true if $f_n \to f$ in μ-measure instead of μ-a.e.

   HINT : observe that $2^{p-1}(|f|^p + |f_n|^p) - |f - f_n|^p \geq 0$ and then use Fatou's lemma.
2. Deduce that $f_n \to f$ in $L^p(\mu)$ if and only if $\|f_n\|_{L^p(\mu)} \to \|f\|_{L^p(\mu)}$ and $f_n \to f$ in μ-measure.

Exercise 10.11 (Uniform integrability)**.** Let $(\Omega, \mathcal{F}, \mu)$ be a measure space with $\mu(\Omega) < \infty$. Let $f_i : \Omega \to \mathbb{R}, i \in I$ be a collection of measurable functions. The collection $(f_i)_{i\in I}$ is called uniformly integrable if

$$\forall\, \varepsilon > 0,\ \exists\, \alpha > 0 : \sup_{i\in I} \int_{\{|f_i|>\alpha\}} |f_i|\, d\mu < \varepsilon, \quad i.e. \quad \lim_{n\to\infty} \sup_{i\in I} \int_{\{|f_i|>n\}} |f_i|\, d\mu = 0.$$

1. Let $f : \Omega \to \mathbb{R}$ be measurable. Justify that the collection consisting of the function f alone is uniformly integrable if and only if $f \in L^1(\Omega, \mu)$. Deduce that any finite collection of integrable functions is uniformly integrable.
2. Show that if there exists $g \in L^1(\Omega, \mu)$ such that $|f_i| \leq g$ for all $i \in I$, then $(f_i)_{i\in I}$ is uniformly integrable.
3. Prove that the collection $(f_i)_{i\in I}$ is uniformly integrable if and only if the two following conditions hold :
   (a) $\sup_{i\in I} \int_\Omega |f_i|\, d\mu < \infty$;
   (b) $\forall\, \varepsilon > 0, \exists\, \delta > 0 : \mu(A) < \delta \Rightarrow \sup_{i\in I} \int_A |f_i| < \varepsilon$.
4. Assume that there exists $\phi : [0, \infty) \to [0, \infty)$ such that $\lim_{r\to\infty} \phi(r)/r = \infty$ and

$$\sup_{i\in I} \int_\Omega \phi(|f_i|)\, d\mu < \infty.$$

   Prove that $(f_i)_{i\in I}$ is uniformly integrable.

   HINT : let $\varepsilon > 0$ and $R > 0$ be such that $\phi(r) \geq r/\varepsilon$ for all $r > R$.

   This criterion of uniform integrability is due to the Belgian mathematician Charles-Jean de la Vallée-Poussin $(1866 - 1962)$.
5. Prove that if the collection $(f_i)_{i\in I}$ is bounded in $L^q(\Omega, \mu)$ for some $q > 1$, then $(|f_i|^p)_{i\in I}$ is uniformly integrable for any $p \in [1, q)$. In particular, $(f_i)_{i\in I}$ is uniformly integrable.

Exercise 10.12 (Lebesgue-Vitali's convergence theorem)**.** Let $(\Omega, \mathcal{F}, \mu)$ be a measure space with $\mu(\Omega) < \infty$. Let $f, f_n : \Omega \to \mathbb{R} \in L^p(\Omega, \mu)$, $n \geq 1$, where $1 \leq p < \infty$. Prove that $f_n \to f$ in $L^p(\Omega, \mu)$ if and only if $f_n \to f$ in μ-measure and the collection $\{|f_n|^p : n \geq 1\}$ is uniformly integrable. To do so, you may proceed as follows :

1. Assume first that $f_n \to f$ in $L^p(\Omega, \mu)$.
   (a) Justify that $f_n \to f$ in μ-measure and that $\int_\Omega |f_n|^p \, d\mu \to \int_\Omega |f|^p \, d\mu$.
   (b) Applying Scheffé's lemma to $g_n := |f_n|^p$, deduce that
   $$\sup_{A \in \mathcal{F}} \left| \int_A |f_n|^p \, d\mu - \int_A |f|^p \, d\mu \right| \overset{n \to \infty}{\to} 0.$$
   (c) Let $\varepsilon > 0$. Recalling that a finite collection of integrable functions is uniformly integrable, show that there exists $\delta > 0$ such that, for all $A \in \mathcal{F}$ with $\mu(A) < \delta$, we have
   $$\sup_{n \geq 1} \int_A |f_n|^p \, d\mu < \varepsilon.$$
   (d) Conclude that $\{|f_n|^p : n \geq 1\}$ is uniformly integrable.
2. Assume now that $f_n \to f$ in μ-measure and that $\{|f_n|^p : n \geq 1\}$ is uniformly integrable.
   (a) Prove that, for any $A \in \mathcal{F}$ and any $n \geq 1$, we have
   $$\int_A |f_n - f|^p \, d\mu \leq 2^{p-1} \int_A |f_n|^p \, d\mu + 2^{p-1} \int_A |f|^p \, d\mu.$$
   Deduce that $\{|f_n - f|^p : n \geq 1\}$ is uniformly integrable.
   (b) Let $M > 0$. Show that, for any $\delta \in (0, M)$ and any $n \geq 1$, we have
   $$\int_{\{|f_n - f| \leq M\}} |f_n - f|^p \, d\mu \leq \delta^p \mu(\Omega) + M^p \mu(\{|f_n - f| > \delta\}).$$
   (c) Conclude that $\int_\Omega |f_n - f|^p \, d\mu \to 0$ as $n \to \infty$.

Exercise 10.13. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space given by $\Omega = [0,1]$, $\mathcal{F} = \mathcal{R}_{[0,1]}$ and $\mathbb{P} = \lambda_{|[0,1]}$. For any random variable $X : \Omega \to \mathbb{R}$ one can adopt either of the following notations :
$$\mathbb{E}[X] = \int_{[0,1]} X(\omega) \, d\mathbb{P}(\omega) = \int_0^1 X(t) \, dt.$$

1. Consider the sequence $(X_n)_{n \geq 1}$ defined by $X_n : \Omega \to \mathbb{R}, \omega \to \sqrt{n}(-\omega)^n$. For which values of $p \geq 1$ does $(X_n)_{n \geq 1}$ converge in $L^p([0,1])$? Does it converge in probability (i.e. in measure), almost everyhere ? Is it uniformly integrable ?
2. Consider the sequence $(Y_n)_{n \geq 1}$ defined by $Y_1 := 1_{[0,1]}$ and $Y_{2^k + j} = \sqrt{2^n} 1_{[j2^{-n}, (j+1)2^{-n}]}$, for all $0 \leq j \leq 2^n - 1$ and $n \geq 1$. For which values of $p \geq 1$ does $(Y_n)_{n \geq 1}$ converge in $L^p([0,1])$? Does it converge in probability, almost everywhere ? Is it uniformly integrable ?
3. Consider the sequence $(Z_n)_{n \geq 1}$ defined by $Z_n = n 1_{[0, \frac{1}{n}]} - n 1_{[1 - \frac{1}{n}, 1]}$. Show that there exists a random variable Z (i.e. a measurable function $Z : \Omega \to \mathbb{R}$) such that $Z_n \to Z$ in probability and $\mathbb{E}[Z_n] \to \mathbb{E}[Z]$, but Z_n does not converge in $L^1([0,1])$. Which assumption of which theorem is not satisfied ?

Exercise 10.14. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and some $p \in [1, \infty)$, consider
$$B_p := \left\{ X \in L^p(\Omega, \mathbb{P}) : \mathbb{E}\big[|X|^p\big]^{1/p} \leq 1 \right\}.$$
Show that B_p is uniformly integrable for $p > 1$ but not for $p = 1$.

Types of convergence and L^p spaces : solutions

Solution de l'exercice 10.1.

(f_n). For any x, we have $f_n(x) \to 0$. In particular, $f_n \to 0$ a.e.. It is clear that for any $a \neq 0$, we don't have

$$\lim_{n\to\infty} \mu(\{|f_n - a| > \varepsilon\}) = 0, \quad \forall\, \varepsilon > 0\cdot$$

For any $\varepsilon \in (0,1)$, we have

$$\mu(\{|f_n| > \varepsilon\}) = 1\cdot$$

It follows that (f_n) does not convergence in measure. We further have that $\|f_n - a\|_{L^\infty(\mathbb{R})} \geq 1$ for any $a \in \mathbb{R}$. For $p \in [1,\infty)$, we also have $\|f_n - a\|_{L^p(\mathbb{R})} = \infty$ if $a \neq 0$ and $\|f_n\|_{L^p(\mathbb{R})} = 1$. It follows that (f_n) does not converge in any $L^p(\mathbb{R})$ norm.

(g_n). We have $f_n \to 0$ a.e.. We also have $f_n \to 0$ in measure since

$$\mu(\{|g_n| > \varepsilon\}) = 1/n,$$

for any $\varepsilon \in (0,1)$, while $\mu(\{|g_n| > \varepsilon\}) = 0$ for any $\varepsilon > 1$. But we have no convergence in any $L^p(\mathbb{R})$. Indeed, for $p = \infty$, we have $\|g_n - a\|_{L^\infty(\mathbb{R})} \geq n - a$. For $p \in [1,\infty)$, we have $\|g_n - a\|_{L^p(\mathbb{R})} = \infty$ for any $a \neq 0$ and $\|g_n\|_{L^p(\mathbb{R})} = 1$.

(h_n). We have $h_n \to 0$ in measure. But no convergence almost everywhere or in $L^p(\mathbb{R})$, $p \in [1,\infty]$, holds. $\square$

Solution de l'exercice 10.2.

Observe that

$$\begin{aligned} f_n \to f \quad \mu\text{−a.e.} &\iff \limsup_{n\to\infty} |f_n - f| = 0 \quad \mu\text{−a.e.} \\ &\iff \mu(\{\limsup_{n\to\infty} |f_n - f| > 0\}) = 0 \\ &\iff \forall \varepsilon > 0 : \mu(\{\limsup_{n\to\infty} |f_n - f| > \varepsilon\}) = 0, \end{aligned}$$

since $\{\limsup_{n\to\infty} |f_n - f| > \varepsilon\} \nearrow \{\limsup_{n\to\infty} |f_n - f| > 0\}$. Let us show that

$$\limsup_{n\geq 1}\{|f_n - f| > 2\varepsilon\} \subset \{\limsup_{n\to\infty} |f_n - f| > \varepsilon\} \subset \limsup_{n\geq 1}\{|f_n - f| > \varepsilon\}.$$

Let $x \in \limsup_{n\geq 1}\{|f_n - f| > 2\varepsilon\}$, i.e. $\forall N \geq 1, \exists k \geq N : |f_k(x) - f(x)| > 2\varepsilon$, i.e. there exists a subsequence (n_ℓ) such that $|f_{n_\ell}(x) - f(x)| > 2\varepsilon$ for all $\ell \geq 1$. It directly follows that $\limsup_{n\to\infty} |f_n(x) - f(x)| > \varepsilon$, i.e. $x \in \{\limsup_{n\to\infty} |f_n - f| > \varepsilon\}$.

Let now $x \in \{\limsup_{n\to\infty} |f_n - f| > \varepsilon\}$, i.e. $\limsup_{n\to\infty} |f_n(x) - f(x)| > \varepsilon$. It follows that there exists a subsequence (n_k) such that $|f_{n_k}(x) - f(x)| > \varepsilon$ for all $k \geq 1$, i.e. $x \in \limsup_{n\geq 1}\{|f_n - f| > \varepsilon\}$.

Therefore, $\mu(\{\limsup_{n\to\infty} |f_n - f| > \varepsilon\}) = 0$ for all $\varepsilon > 0$ if and only if $\mu(\limsup_{n\geq 1}\{|f_n - f| > \varepsilon\})$ for every $\varepsilon > 0$. This concludes the proof. $\square$

Solution de l'exercice 10.3.

From exercise 10.2, we have that $f_n \to f$ μ-a.e. if and only if $\mu(\limsup_{n\geq 1}\{|f_n - f| > \varepsilon\}) = 0$ for every $\varepsilon > 0$. But

$$\mu(\limsup_{n\geq 1}\{|f_n - f| > \varepsilon\}) = \mu\Bigg(\bigcap_{n\geq 1}\bigcup_{k\geq n}\{|f_k - f| > \varepsilon\}\Bigg) = \lim_{n\to\infty}\mu\Bigg(\bigcup_{k\geq n}\{|f_k - f| > \varepsilon\}\Bigg),$$

provided that continuity of measures from above applies (the sets descrease in n), i.e. if we can guarantee that $\mu\big(\bigcup_{k\geq n}\{|f_k - f| > \varepsilon\}\big) < \infty$ for some n. This is the case in both the hypotheses of points 1 and 2.

3. Even though this result has already been proved during the lectures, we are now able to provide a new proof. If $f_n \to f$ μ-a.e. and $\mu(X) < \infty$, we have by point 2 that $\lim_{n\to\infty}\mu\big(\bigcup_{k\geq n}\{|f_k - f| > \varepsilon\}\big) = 0$ for every $\varepsilon > 0$. In particular

$$\mu(\{|f_n - f| > \varepsilon\}) \leq \mu\Bigg(\bigcup_{k\geq n}\{|f_k - f| > \varepsilon\}\Bigg) \overset{n\to\infty}{\to} 0.$$

We conclude that $f_n \to f$ in measure. $\square$