Recovering a probability measure from its multivariate spatial rank

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Abstract: We address the problem of recovering a probability measure P over $\mathbb{R}^n$ from the associated multivariate spatial rank R_P. Multivariate spatial ranks characterize probability measures (see [14]). If P has a density f_P, we strengthen this result and show that $f_P = \mathscr{L}_n(R_P)$, where $\mathscr{L}_n$ is a (potentially fractional) partial differential operator given in closed form and depends on n. When P admits no density, we further show that the equality $P = \mathscr{L}_n(R_P)$ still holds in the sense of distributions (i.e. generalized functions). We thoroughly investigate the regularity properties of spatial ranks and use the PDE we established to give qualitative results on depths contours and regions. We study the local properties of the operator $\mathscr{L}_n$ and show that it is non-local when n is even. We conclude the paper with a partial counterpart to the non-localizability in even dimensions.

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Contents

1. Introduction

The cumulative distribution function (cdf) and quantiles of a probability measure over $\mathbb{R}$ play a central role in probability and statistics. They characterize the underlying probability measure and allow one to control the probability content of a given region. One of the most notable use of quantiles is arguably the design of hypotheses tests in which one is able to establish confidence regions for the estimation of a parameter of interest. Furthermore, the cdf pushes the probability measure from which it derives onto the uniform distribution over the interval $[0, 1]$. This gives a very simple way to generate random observations that follow a given probability law. For these reasons, a lot of effort has been made over the past decades to extend the notions of cdf and quantiles of a probability measure to a multivariate setting. It has long been known that considering quantiles of a probability measure P over $\mathbb{R}^n$ componentwise leads to quantile regions that are not equivariant with respect to orthogonal transformations, which automatically discarded this approach. The first and probably most famous attempt to establish a multivariate analog of cumulative distribution functions and quantiles is the concept of *halfspace depth*, introduced in [31], which, to any point of $\mathbb{R}^n$, associate a non-negative number (its depth). The regions of points the depth of which does not exceed a given treshold value are interpreted as analogs of quantile regions in $\mathbb{R}$. Many other concepts have followed, such as simplicial depth [18] and projection depth [32] to cite only a few. Other approaches have been adopted, attempting to define a proper notion of multivariate quantiles and cumulative distribution functions (often called *ranks*), some of the most notable being based on regression quantiles [13] or optimal transport [12]. We also refer the reader to [27] for a review on the topic.

Among the concepts extending quantiles and cumultative distribution functions to a multivariate setting, one very popular approach is that of *geometric* or *spatial* multivariate ranks and quantiles, introduced in [5]. Spatial ranks and quantiles enjoy important advantages over other competing approaches. Among them, let us stress that spatial ranks are available in closed forms which leads to trivial evaluation in the empirical case, unlike most competing concepts. As a consequence, explicit Bahadur-type representations and asymptotic normality results are provided in [5] and [14], when competing approaches offer at best consistency results only. Spatial ranks and quantiles also allow for direct extensions in infinite-dimensional Hilbert spaces ; see, e.g., [4] and [6]. We refer the reader to [23] for an overview of the scope of applications spatial ranks offer.

Let P be a probability measure P over $\mathbb{R}^n$. A *spatial quantile* of order $\alpha \in [0, 1)$ in direction $u \in S^{n-1}$ for P is defined as an arbitrary minimizer of the objective function

$$x \mapsto O^P_{\alpha,u}(x) := \int_{\mathbb{R}^n} \Big\{ |z - x| - |z| - (\alpha u, x) \Big\}\, dP(z),$$

where $|y| := \sqrt{(y, y)}$ is the Euclidean norm of $y \in \mathbb{R}^n$ and $(u, v) = \sum_{i=1}^n u_i v_i$ is the Euclidean inner product between $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$ and $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$. Let also

$$R_P(x) = \int_{\mathbb{R}^n \setminus \{z\}} \frac{x - z}{|x - z|}\, dP(z)$$

for any $x \in \mathbb{R}^n$ denote the *spatial rank* map of P. In the univariate case $n = 1$, it is showed in [8] that spatial quantiles of order $\alpha \in [0, 1)$ in direction $u \in \{-1, +1\}$ reduce to the usual quantiles of order $(\alpha u + 1)/2 \in [0, 1)$. Still when $n = 1$, we have that

$$R_P(x) = \int_{\mathbb{R}} \operatorname{sign}(x - z)\, dP(z)$$

and

$$F_P(x) = 2R_P(x) - 1$$

for any $x \in \mathbb{R}$, where $F_P(x) := P[(-\infty, x]]$ stands for the usual univariate cumulative distribution function of P. This motivates calling Q_P and R_P multivariate analogs of the univariate quantile map and univariate cumulative distribution function, respectively.

In a general multivariate framework $n \geq 1$ it is proved in [15] that $O^P_{\alpha,u}$ is continuously differentiable over $\mathbb{R}^n$ and that

$$\nabla O^P_{\alpha,u}(x) = R_P(x) - \alpha u$$

for any $x \in \mathbb{R}^n$, provided P has no atoms. Further requiring that P is not supported on a single line of $\mathbb{R}^n$, it is proved in [25] that $O^P_{\alpha,u}$ is strictly convex over $\mathbb{R}^n$ and, therefore, that spatial quantiles of order α in direction u for P are unique for any $\alpha \in [0,1)$ and any $u \in S^{n-1}$; let us denote such a quantile by $Q_P(\alpha u)$. This implies that $Q_P(\alpha u)$, i.e. the unique minimizer of $O^P_{\alpha,u}$, is the unique solution $x \in \mathbb{R}^n$ of the equation

$$R_P(x) = \alpha u.$$

Under the above assumptions, it also showed in [15] that the quantile map $\alpha u \mapsto Q_P(\alpha u)$ is invertible with inverse given by R_P. This provides another motivation to regard R_P as a natural multivariate analog of the univariate cumulative distribution function.

Of particular interest are the so-called *depth contours* and *depth regions* of P. Assume that P is not supported on a single line of $\mathbb{R}^n$ so that spatial quantiles for P are unique. For any $\beta \in [0,1)$, we let the depth region $\mathcal{D}^\beta_P$ and the depth contour $\mathcal{C}^\beta_P$ of P of order β be defined as

$$\mathcal{D}^\beta_P = \Big\{ Q_P(\alpha u) : \alpha \in [0,\beta],\ u \in S^{n-1} \Big\}$$

and

$$\mathcal{C}^\beta_P = \Big\{ Q_P(\beta u) : u \in S^{n-1} \Big\}.$$

Depth regions provide a family of smooth compact arc-connected and nested centrality regions, while depth contours are disjoint compact and arc-connected $(n-1)$-dimensional smooth manifolds (see Section 7).

As we already mentioned, the conceptual and computational simplicity of spatial ranks and quantiles allow for explicit qualitative and quantitative results. Therefore, spatial ranks and quantiles are really well-understood; see, e.g. [10] and [11] for interesting features of spatial quantiles. Similarly to their univariate counterpart, it is well-know that spatial ranks characterize probability measures in arbitrary dimension n : if P and Q are Borel probability measures over $\mathbb{R}^n$ and if $R_P(x) = R_Q(x)$ for any $x \in \mathbb{R}^n$, then $P = Q$ (see Theorem 2.5 in [14]). Note that this very desirable property is also shared by ranks based on optimal transport and, when P admits a sufficiently smooth density, the density can be recovered from the rank via a (highly non-linear) partial differential equation, see [12]. The characterization property is not shared by the concept of halfspace depth (see [20]). However, halfspace depth possesses the characterization property within an important class of probability measures ; see [30] who gave the first positive result in the case of empirical probability measures by algorithmically reconstructing the measure. We refer the reader to [21] for a review on the question of characterization for halfspace depth. Therefore, it is most natural to address the question of the possibility of recovering a probability measure from its spatial rank. In the present paper, we show that any Borel probability measure P over $\mathbb{R}^n$ can be reconstructed from its spatial rank R_P through a linear (potentially fractional) partial differential equation involving R_P only (even when P admits no density), extending the characterization result from [14] with a degree of generality that outperforms similar results known for halfspace-depth and quantiles based on optimal transport.

The structure of the paper is as follows. In section 2, the main definitions are stated. We discuss the strategy used in the paper to recover a probability measure P over $\mathbb{R}^n$ knowing its multivariate spatial rank R_P only. Some notations and usual spaces are introduced in Section 3. Section 4 is devoted to a brief review on distribution theory and Sobolev spaces. We introduce fractional Laplacians, which are key ingredients all along the paper, in Section 5. In Section 6, we establish the PDE relating an arbitrary probability measure P over $\mathbb{R}^n$ to its multivariate spatial rank R_P in the sense of distributions. We thoroughly investigate the regularity properties of spatial ranks and give sufficient conditions for the above PDE to hold pointwise. We devote Section 7 to establishing some regularity properties of quantile regions and quantile contours by exploiting the regularity results obtained in Section 6 for spatial ranks. In Section 8 we give a refinement in odd dimensions of the characterization property of spatial ranks given in Theorem 2.5 of [14]. We also give a partial counterpart to the non-local nature of the PDE in even dimensions.

2. Main results

Throughout, we will write $\mathbb{I}[A]$ for the indicator function of the condition A.

In the following definition, we introduce a key quantity that is strongly related to the map $O^P_{\alpha,u}$ introduced in Section 1.

Definition 2.1. *Let $n \geq 1$ and P be a Borel probability measure over $\mathbb{R}^n$. We define the map $g_P : \mathbb{R}^n \to \mathbb{R}$ by letting*

$$g_P(x) = \mathbb{E}\big[|x - Z| - |Z|\big]$$

for any $x \in \mathbb{R}^n$, where Z is a random n-vector with law P.

We obivously have that $O^P_{\alpha,u}(x) = g_P(x) - (\alpha u, x)$ for any $x \in \mathbb{R}^n$, so that

$$\nabla O^P_{\alpha,u} = \nabla g_P - \alpha u$$

whenever $O^P_{\alpha,u}$ is differentiable. In view of the discussion of Section 1, it is clear that g_P is strongly related to R_P.

The triangle inequality yields that g_P is well-defined, irrespective of the probability measure P : no moment assumption is made. It is further easy to see that g_P is continuous over $\mathbb{R}^n$. Theorem 5.6 in [15] entails that g_P is continuously differentiable over an open subset $U \subset \mathbb{R}^n$ if and only if P has no atoms over U. In that case, we have that $R_P(x) = \nabla g_P(x)$ for any $x \in U$, where R_P is given in the next definition.

Definition 2.2. *Let $n \geq 1$ and P be a Borel probability measure over $\mathbb{R}^n$. The spatial rank R_P of P is the map $R_P : \mathbb{R}^n \to \mathbb{R}^n$ defined by*

$$R_P(x) = \mathbb{E}\left[\frac{x - Z}{|x - Z|}\mathbb{I}[Z \neq x]\right]$$

for any $x \in \mathbb{R}^n$, where Z is a random n-vector with law P.

The reason why we introduced g_P and stressed the fact that $\nabla g_P = R_P$ will appear later on.

In order to recover P from the knowledge of R_P only, it is crucial to notice the key fact that R_P is a convolution between P and a fixed kernel, which we will denote by K in the sequel. Formally taking the Fourier transform $\mathscr{F}(R_P)$ of R_P (in the sense of distributions, i.e. generalized functions) then gives

$$\mathscr{F}(R_P) = \mathscr{F}(K)\mathscr{F}(P). \tag{1}$$

This fact was already noticed in [14] and we used this idea as a building block to prove the results we present in this paper. Recovering P now essentially amounts to isolating $\mathscr{F}(P)$ in (1) and then taking the inverse Fourier transform of $\mathscr{F}(P)$. Obtaining an explicit formula for $\mathscr{F}(P)$ will however require having an explicit formula for $\mathscr{F}(K)$ in return. Let us then introduce the kernel $K : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$K(x) = \begin{cases} \frac{x}{|x|} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

In short, we write $K(x) = \frac{x}{|x|}\mathbb{I}[x \neq 0]$. We formally write $R_P(x) = (K * P)(x)$. We will sometimes write K_n instead of K to stress the fact that we consider the kernel K in $\mathbb{R}^n$ when confusion about the dimension is possible.

We will show that the Fourier transform $\mathscr{F}(K_n)$ of K_n is given, up to a multiplicative constant C_n that only depends on n, by

$$\text{P.V.}\Big(\frac{\xi}{|\xi|^{n+1}}\Big),$$

where P.V. denotes the principal value. In other words, we have that

$$\int_{\mathbb{R}^n} K_n(x)\widehat{\psi}(x)\,dx = C_n \lim_{\eta\to 0}\int_{\mathbb{R}^n\setminus B_\eta}\frac{\xi}{|\xi|^{n+1}}\psi(\xi)\,d\xi$$

for any $\psi \in \mathscr{S}(\mathbb{R}^n)$, where B_η stands for the ball of radius $\eta > 0$ centered at the origin and $\mathscr{S}(\mathbb{R}^n)$, or $\mathscr{S}(\mathbb{R}^n, \mathbb{C})$, stands for the complex-valued Schwarz class over $\mathbb{R}^n$. Before stating the main result of the paper, we introduce the operator $\mathscr{L}_n$, that will play a key role in this paper. It involves a constant γ_n defined by

$$\frac{1}{\gamma_n} = 2^n\pi^{\frac{n-1}{2}}\Gamma\Big(\frac{n+1}{2}\Big).$$

The domain $D(\mathscr{L}_n)$ of the operator $\mathscr{L}_n$ is a space of distributions that depends on n, namely we let $D(\mathscr{L}_n) = \mathscr{S}^n(\mathbb{R}^n)'$, the space of $\mathbb{C}^n$-valued tempered distributions (see Section 4.1 for further details), if n is odd and $D(\mathscr{L}_n) = \mathscr{S}^n_{1/2}(\mathbb{R}^n)'$ (see Section 5) if n is even.

Definition 2.3 (The operator $\mathscr{L}_n$ and its adjoint $\mathscr{L}_n^*$)**.** *Let $n \in \mathbb{N}$ with $n \geq 1$. Let us define the (potentially fractional) differential operator $\mathscr{L}_n : D(\mathscr{L}_n) \subset \mathscr{S}^n(\mathbb{R}^n)' \to \mathscr{S}(\mathbb{R}^n)'$ by*

$$\mathscr{L}_n := \gamma_n \begin{cases} (-\Delta)^{\frac{n-1}{2}}\nabla\,\cdot & \text{if } n \text{ is odd,}\\ (-\Delta)^{\frac{1}{2}}(-\Delta)^{\frac{n-2}{2}}\nabla\,\cdot & \text{if } n \text{ is even,}\end{cases}$$

where $\nabla\,\cdot$ stands for the divergence operator, $(-\Delta)^k$ stands for the Laplacian $-\Delta$ taken k times successively when $k \in \mathbb{N}$, and $(-\Delta)^{\frac{1}{2}}$ is the fractional Laplacian introduced in Section 5. Let us define the formal adjoint $\mathscr{L}_n^* : D(\mathscr{L}_n^*) \subset \mathscr{S}(\mathbb{R}^n)' \to \mathscr{S}^n(\mathbb{R}^n)'$ of $\mathscr{L}_n$ by

$$\mathscr{L}_n^* := \gamma_n \begin{cases} \nabla(-\Delta)^{\frac{n-1}{2}} & \text{if } n \text{ is odd,}\\ \nabla(-\Delta)^{\frac{1}{2}}(-\Delta)^{\frac{n-2}{2}} & \text{if } n \text{ is even,}\end{cases}$$

where ∇ stands for the gradient operator and

$$D(\mathscr{L}_n^*) = \begin{cases} \mathscr{S}(\mathbb{R}^n)' & \text{if } n \text{ is odd,}\\ \mathscr{S}_{1/2}(\mathbb{R}^n)' & \text{if } n \text{ is even.}\end{cases}$$

We call $\mathscr{L}_n^*$ the formal adjoint of $\mathscr{L}_n$ since we have

$$\langle \mathscr{L}_n\Lambda, \varphi\rangle = \langle \Lambda, \mathscr{L}_n^*\varphi\rangle$$

for any $\Lambda \in D(\mathscr{L}_n)$ and any $\varphi \in \mathscr{S}(\mathbb{R}^n)$, and

$$\langle \mathscr{L}_n^* T, \Psi\rangle = \langle T, \mathscr{L}_n\Psi\rangle$$

for any $T \in D(\mathscr{L}_n^*)$ and any $\Psi \in \mathscr{S}(\mathbb{R}^n, \mathbb{C}^n)$. In particular, we have

$$\int_{\mathbb{R}^n}(\mathscr{L}_n\Psi)(x)\varphi(x)\,dx = \int_{\mathbb{R}^n}(\Psi(x), \mathscr{L}_n^*(\varphi)(x))\,dx$$

for any $\Psi \in \mathscr{S}(\mathbb{R}^n, \mathbb{C}^n)$ and any $\varphi \in \mathscr{S}(\mathbb{R}^n)$.

Taking Fourier transforms shows that all differential operators involved in the definition of $\mathscr{L}_n$ and $\mathscr{L}_n^*$ commute with each other over $D(\mathscr{L}_n)$ and $D(\mathscr{L}_n^*)$, respectively. This legitimates writing $\mathscr{L}_n$ and $\mathscr{L}_n^*$ in the more compact forms

$$\mathscr{L}_n = \gamma_n(-\Delta)^{\frac{n-1}{2}}\nabla\,\cdot$$

and

$$\mathscr{L}_n^* = \gamma_n\nabla(-\Delta)^{\frac{n-1}{2}},$$

irrespective of $n \geq 1$. In $\mathbb{R}$, notice that $\mathscr{L}_1$ and $\mathscr{L}_1^*$ simply reduce to

$$\mathscr{L}_1 = \frac{1}{2}\frac{d}{dx} = \mathscr{L}_1^*.$$

From there, we are able to deduce the following result.

Theorem 2.4. *Let $n \geq 1$ and P be a Borel probability measure over $\mathbb{R}^n$. Then R_P belongs to the domain $D(\mathscr{L}_n)$ of $\mathscr{L}_n$ and the equality*

$$P = \mathscr{L}_n(R_P)$$

holds in $\mathscr{S}(\mathbb{R}^n)'$, i.e.

$$\int_{\mathbb{R}^n}\psi(x)\,dP(x) = \int_{\mathbb{R}^n}\Big(R_P(x), (\mathscr{L}_n^*\psi)(x)\Big)\,dx$$

for any $\psi \in \mathscr{S}(\mathbb{R}^n)$.

When P admits a density f_P over $\mathbb{R}^n$, we want to use the previous result to state a pointwise equality between f_P and $\mathscr{L}_n(R_P)$. When n is odd, R_P therefore needs to be at least n times differentiable. When n is even, R_P need to be at least $n-1$ times differentiable and such that $(-\Delta)^{\frac{1}{2}}(-\Delta)^{\frac{n-2}{2}}(\nabla\cdot R_P)$ makes sense pointwise. Differentiability up to order $n-1$ is obtained in Proposition 6.8 by a standard dominated convergence argument. Since $\frac{1}{|x|^n}$ is not integrable near the origin nor at infinity in $\mathbb{R}^n$, reaching differentiability of order n must be done differently since the $(n-1)$th derivatives of R_P behave like

$$x \mapsto \int_{\mathbb{R}^n}\frac{1}{|x-z|^{n-1}}f_P(z)\,dz.$$

Our strategy consists in using the PDE we established, i.e.

$$f_P = \gamma_n(-\Delta)^{\frac{n-1}{2}}(\nabla\cdot R_P)$$

and use elliptic regularity to deduce that $\nabla\cdot R_P \in H^{n-1}_{\text{loc}}$. But this fact alone is not enough to conclude that $R_P \in H^n_{\text{loc}}$. This is where the identity $R_P = \nabla g_P$ plays a crucial role. Indeed, we will therefore have that $-\Delta g_P \in H^{n-1}_{\text{loc}}$. This will lead to $g_P \in H^{n+1}_{\text{loc}}$ whence $R_P \in H^n_{\text{loc}}$.

Theorem 2.5 (Odd dimensions)**.** *Let $n \geq 3$ be odd and P be a Borel probability measure over $\mathbb{R}^n$. Let $\Omega \subset \mathbb{R}^n$ be an open subset and assume that P admits a density $f_\Omega \in L^1(\Omega)$ over Ω with respect to the Lebesgue measure. Then, we have the following :*

1. *If $f_\Omega \in L^p_{\text{loc}}(\Omega)$ for some $p > n$, then $R_P \in \mathscr{C}^{n-1}(\Omega) \cap H^n_{\text{loc}}(\Omega)$. In particular, R_P admits weak derivatives of order n in Ω and we have that*

$$f_\Omega(x) = (\mathscr{L}_nR_P)(x)$$

for almost any $x \in \Omega$.

2. *If $f_\Omega \in \mathscr{C}^{k,\alpha}_{\text{loc}}(\Omega)$ for some $k \in \mathbb{N}$ and some $\alpha \in (0,1)$, then $R_P \in \mathscr{C}^{k+n}(\Omega)$ and we have that*

$$f_\Omega(x) = (\mathscr{L}_nR_P)(x)$$

for any $x \in \Omega$.

Theorem 2.6 (Even dimensions)**.** *Let $n \geq 2$ be even and P be a Borel probability measure over $\mathbb{R}^n$. Assume that P admits a density $f_P \in L^1(\mathbb{R}^n)$ with respect to the Lebesgue measure. Further let $R_P^{(n-1)} := \gamma_n(-\Delta)^{\frac{n-2}{2}}(\nabla \cdot R_P)$. Then, we have the following :*

1. *If $n > 2$ and if $f_P \in L^p_{\text{loc}}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ for some $p > n$, then $R_P \in \mathscr{C}^{n-1}(\mathbb{R}^n) \cap H^n_{\text{loc}}(\mathbb{R}^n)$. In particular, R_P admits weak derivatives of order n in $\mathbb{R}^n$ and we have that*

$$f_P(x) = (\mathscr{L}_n R_P)(x) = ((-\Delta)^{\frac{1}{2}} R_P^{(n-1)})(x)$$

for almost any $x \in \mathbb{R}^n$. In addition, $R_P^{(n-1)} \in H^1(\mathbb{R}^n)$ and we have that

$$((-\Delta)^{\frac{1}{2}} R_P^{(n-1)})(x) = 2\pi \mathscr{F}^{-1}\Big(|\xi| \mathscr{F} R_P^{(n-1)}\Big)(x)$$

for almost any $x \in \mathbb{R}^n$.

2. *If $f_P \in \mathscr{C}^{k,\alpha}(\mathbb{R}^n)$ for some $k \in \mathbb{N}$ and some $\alpha \in (0,1)$, then $R_P \in \mathscr{C}^{k+n}(\mathbb{R}^n)$ and we have that*

$$f_P(x) = (\mathscr{L}_n R_P)(x)$$

for any $x \in \mathbb{R}^n$. In addition, we have that

$$(\mathscr{L}_n R_P)(x) = c_{n,1/2} \lim_{\eta \to 0} \int_{\mathbb{R}^n \setminus B_\eta(x)} \frac{R_P^{(n-1)}(x) - R_P^{(n-1)}(z)}{|x-z|^{n+1}}\, dz$$

for any $x \in \mathbb{R}^n$ (see Section 5 for the definition of $c_{n,1/2}$).

The fundamental difference between odd and even dimensions lies in the local nature of the statement in odd dimensions and the global nature of the statement in even dimensions. This is a direct consequence of the fact that $\mathscr{L}_n$ is a purely local differential operator when n is odd, while $\mathscr{L}_n$ is non-local when n is even due to the presence of $(-\Delta)^{\frac{1}{2}}$ in $\mathscr{L}_n$.

In even dimensions, we require $f_P \in L^2(\mathbb{R}^n)$ although we do not ask it in odd dimensions. This is again due to the local nature of the statement in odd dimensions, which actually requires $f_P \in L^2_{\text{loc}}$. But this condition is automatically verified since we already asked $f_P \in L^p_{\text{loc}}$ with $p > n \geq 2$. Now, the fact that we require $f_P \in L^2(\mathbb{R}^n)$ in even dimensions implies that $\frac{1}{|x|^{n-1}}$ does not belong to $L^2(\mathbb{R}^n)$ when $n = 2$. This is why $n = 2$ is excluded in even dimensions if no Hölder regularity holds.

3. Notations

Let $n \geq 1$ and $U \subset \mathbb{R}^n$ be an open subset.

- We let $\mathbb{N} = \{0, 1, 2, \ldots\}$ stand for the collection of non-negative integers.
- We will denote the inner product over $\mathbb{R}^n$ by $(\cdot, \cdot)$.
- For any $x \in \mathbb{R}^n$ and $r > 0$, we let $B_r(x)$ and B_r denote the open ball centered at x with radius r and the ball centered at the origin with radius r, respectively.
- For any subset $A \subset \mathbb{R}^n$, we let $\overline{A}$ denote the closure of A with respect to the usual topology of $\mathbb{R}^n$.
- For a function $u : \mathbb{R}^n \to \mathbb{C}$ that is k-times differentiable, we let

$$\partial^\alpha u(x) := \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}(x)$$

  for any $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ such that $|\alpha| := \sum_{j=1}^n \alpha_j \leq k$. By convention, we let $\partial^\alpha u := u$ if $\alpha = (0, \ldots, 0)$.
- For any $k \geq 0$, we let $\mathscr{C}^k(U)$ stand for the collection of functions $u : U \to \mathbb{C}$ that are k-times differentiable and such that $\partial^\alpha u$ is continuous over U for any $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq k$.

- For any $k \geq 0$, we let $\mathscr{C}_b^k(U)$ denote the collection of functions $u \in \mathscr{C}^k(U)$ such that $\partial^\alpha u$ is bounded over U for any $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq k$.
- The set $\mathscr{C}_c^k(U)$ will stand for the subset of $\mathscr{C}_b^k$ made of functions whose support is compact and contained in U. The set $\mathscr{C}_c^\infty(U)$ of infinitely differentiable maps with compact support in U is also denoted $\mathscr{D}(U)$.
- The set $\mathscr{C}_0(\mathbb{R}^n)$ will denote the collections of (complex-valued) continuous functions that converge to 0 at infinity.
- The Hölder space $\mathscr{C}^{k,\alpha}(U)$, $k \in \mathbb{N}$ and $\alpha \in (0,1]$, will denote the collection of functions $u \in \mathscr{C}^k(U)$ such that $\partial^\beta u$ is bounded over U for any $\beta \in \mathbb{N}^n$ with $|\beta| \leq k$ and such that $\partial^\beta u$ is α-Hölder continuous over U when $|\beta| = k$. Similarly, we let $\mathscr{C}^{k,\alpha}_{\text{loc}}(U)$ be the collections of functions $u \in \mathscr{C}^k(U)$ such that $u \in \mathscr{C}^{k,\alpha}(V)$ for any open bounded subset $V \subset U$ such that $\overline{V} \subset U$.
- For any $u \in L^1(\mathbb{R}^n)$, we define the Fourier transform $\widehat{u}$ of u by

$$\widehat{u}(\xi) = \int_{\mathbb{R}^n} u(x) e^{-2i\pi (x,\xi)}\, dx$$

  for any $\xi \in \mathbb{R}^n$. We let $\mathscr{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ denote the Fourier transform on $L^2(\mathbb{R}^n)$, defined as the unique continuous extension of the Fourier transform restricted to $\mathscr{S}(\mathbb{R}^n)$. We will also denote by $\mathscr{F}$ the Fourier transform acting on the space $\mathscr{S}(\mathbb{R}^n)'$ of tempered distributions over $\mathbb{R}^n$ (see Section 4.1), and acting componentwise on $\mathscr{S}^k(\mathbb{R}^n)'$, for any $k \geq 1$.

In the sequel, we will refer to the following statement as Green's formula, that can be found in Appendix C.2 of [7].

Theorem 3.1 (Green's formula). *Let $\Omega \subset \mathbb{R}^n$ be a regular and bounded open subset. Let $u, v \in \mathscr{C}^1(\overline{\Omega})$. For any $i \in \{1, \dots, n\}$, we have*

$$\int_\Omega (\partial_i u(x)) v(x)\, dx = \int_{\partial\Omega} u(x) v(x) \nu_i(x)\, d\sigma(x) - \int_\Omega u(x) \partial_i v(x)\, dx,$$

where $\nu_i(x)$ is the ith component of the outer unit normal vector to Ω at x and σ the surface area measure on $\partial\Omega$.

4. Brief review of distribution theory and Sobolev spaces

In this section, we review some of the basic analytical tools used in the paper : distribution theory and Sobolev spaces.

4.1. Distributions

The reference we used for this section is [26].

Let $U \subset \mathbb{R}^n$ be an open subset. Let us recall that the set of infinitely differentiable functions whose support is compact and included in U is denoted in the paper by $\mathscr{C}_c^\infty(U)$. We endow $\mathscr{C}_c^\infty(U)$ with the following notion of convergence. A sequence $(\varphi_k) \subset \mathscr{C}_c^\infty(U)$ converges to $\varphi \in \mathscr{C}_c^\infty(U)$ in the space $\mathscr{C}_c^\infty(U)$ if there exists a compact subset $K \subset U$ such that $\text{supp}(\varphi_k) \subset K$ for any k and such that

$$\sup_{x \in K} |\partial^\alpha (\varphi_k - \varphi)(x)| \to 0$$

as $k \to \infty$ for any $\alpha \in \mathbb{N}^n$.

Definition 4.1 (Distribution). *A distribution on U is a linear map*

$$T : \mathscr{C}_c^\infty(U) \to \mathbb{C},\ \ \varphi \mapsto \langle T, \varphi \rangle$$

which is continuous with respect to the convergence on $\mathscr{C}_c^\infty(U)$. The set of all distributions on U is denoted $\mathscr{D}(U)'$.

Examples 4.2. We list here typical examples of distributions and a few usual ways to obtain distributions from other distributions.

- Any function $f \in L^1_{\mathrm{loc}}(U)$ gives rise to a distribution T_f on U which, by an obivous abuse of notation we also write f, by letting
$$\langle f, \varphi \rangle := \int_U f(x)\varphi(x)\,dx$$
for any $\varphi \in \mathscr{C}_c^\infty(U)$.
- Similarly, any Borel measure μ on U that is finite over compact subsets of U leads to a distribution on U by letting
$$\langle \mu, \varphi \rangle := \int_U \varphi(x)\,d\mu(x)$$
for any $\varphi \in \mathscr{C}_c^\infty(U)$. In particular, any Borel probability measure is a distribution on any open subset of $\mathbb{R}^n$.
- If $T \in \mathscr{D}(U)'$ is a distribution on U, we define its distributional derivatives $\partial^\alpha T$, $\alpha \in \mathbb{N}^n$, by letting
$$\langle \partial^\alpha T, \varphi \rangle := (-1)^{|\alpha|}\,\langle T, \partial^\alpha \varphi \rangle$$
for any $\varphi \in \mathscr{C}_c^\infty(U)$.
- For any smooth function $f \in \mathscr{C}^\infty(U)$ and any distribution $T \in \mathscr{D}(U)'$ on U, we define the distribution fT by letting
$$\langle fT, \varphi \rangle := \langle T, f\varphi \rangle$$
for any $\varphi \in \mathscr{C}_c^\infty(U)$.

Therefore, distributions are stable with respecto to multiplication by a smooth function and taking derivatives. Other common operations, such as convolution and taking the Fourier transform, do not leave the space $\mathscr{C}_c^\infty(U)$ invariant and therefore cannot be directly defined on $\mathscr{D}(U)'$. Consequently, another class of test functions needs to be used to apply such operations on distributions. This is the essence of tempered distributions, that rely on the Schwarz class $\mathscr{S}(\mathbb{R}^n)$ defined as

$$\mathscr{S}(\mathbb{R}^n) = \Big\{ f \in \mathscr{C}^\infty(\mathbb{R}^n) : \sup_{x \in \mathbb{R}^n} (1+|x|)^m |\partial^\alpha f(x)| < \infty,\ \forall m \in \mathbb{N},\ \forall \alpha \in \mathbb{N}^n \Big\}.$$

It should be noted that functions from $\mathscr{S}(\mathbb{R}^n)$ are complex-valued, in accordance with our definition of $\mathscr{C}^\infty(\mathbb{R}^n)$, see Section 3.

The set $\mathscr{S}(\mathbb{R}^n)$ is a vector space. It is also stable by multiplication (it is an algebra), multiplication by smooth functions for which all derivatives have at most polynomial growth at infinity, convolution, derivation and Fourier transform. We further have the inclusion

$$\mathscr{C}_c^\infty(U) \subset \mathscr{S}(\mathbb{R}^n)$$

for any open subset $U \subset \mathbb{R}^n$. The set $\mathscr{S}(\mathbb{R}^n, \mathbb{C}^k)$, with $k \geq 1$, will stand for the collection of vector fields $\Psi = (\psi_1, \ldots, \psi_k)$ for which every component ψ_i belongs to $\mathscr{S}(\mathbb{R}^n)$. Similarly to the space $\mathscr{C}_c^\infty(\mathbb{R}^n)$, we endow $\mathscr{S}(\mathbb{R}^n)$ with an adequate notion of convergence. A sequence $(\psi_k) \subset \mathscr{S}(\mathbb{R}^n)$ converges to $\psi \in \mathscr{S}(\mathbb{R}^n)$ in the space $\mathscr{S}(\mathbb{R}^n)$ if

$$\sup_{x \in \mathbb{R}^n} |(1+|x|)^m \partial^\alpha (\psi_k - \psi)(x)| \to 0$$

as $k \to \infty$ for any $m \in \mathbb{N}$ and any $\alpha \in \mathbb{N}^n$.

Definition 4.3 (Tempered distribution)**.** *A tempered distribution is a linear map*

$$T : \mathscr{S}(\mathbb{R}^n) \to \mathbb{C},\ \psi \mapsto \langle T, \psi \rangle$$

which is continuous with respect to the convergence on $\mathscr{S}(\mathbb{R}^n)$. The set of all tempered distributions is denoted $\mathscr{S}(\mathbb{R}^n)'$.

For the sake of simplicity we let

$$\mathscr{S}^k(\mathbb{R}^n)' := (\mathscr{S}(\mathbb{R}^n)')^k$$

for any $k \in \mathbb{N}$ with $k \geq 1$ be the set of linear maps

$$T = (T_1, \dots, T_k) : \mathscr{S}(\mathbb{R}^n, \mathbb{C}^k) \to \mathbb{C}^k, \ \Psi = (\psi_1, \dots, \psi_k) \to (\langle T_1, \psi_1 \rangle, \dots, \langle T_k, \psi_k \rangle)$$

such that $T_i \in \mathscr{S}(\mathbb{R}^n)'$ for any $i = 1, \dots, k$. We let all operations described above (multiplication by a smooth function, derivation, Fourier transform and convolution) act on $\mathscr{S}^k(\mathbb{R}^n)'$ componentwise. Therefore, all stated identities remain valid on $\mathscr{S}^k(\mathbb{R}^n)'$.

It is easy to see that if a sequence $(\varphi_k) \subset \mathscr{C}_c^\infty(\mathbb{R}^n)$ converges to some $\varphi \in \mathscr{C}_c^\infty(\mathbb{R}^n)$ in the space $\mathscr{C}_c^\infty(\mathbb{R}^n)$, then convergence also holds in the space $\mathscr{S}(\mathbb{R}^n)'$. In particular, every tempered distribution is a distribution over $\mathbb{R}^n$.

Examples 4.4. We list here typical examples of tempered distributions and a few usual ways to obtain tempered distributions from other tempered distributions.

- If $f : \mathbb{R}^n \to \mathbb{C}$ is a measurable map such that

$$\frac{f(x)}{(1 + |x|)^m} \in L^p(\mathbb{R}^n)$$

  for some $m \in \mathbb{N}$ and some $p \in [1, \infty)$, then $f \in \mathscr{S}(\mathbb{R}^n)'$ by letting

$$\langle f, \psi \rangle := \int_{\mathbb{R}^n} f(x)\psi(x)\, dx$$

  for any $\psi \in \mathscr{S}(\mathbb{R}^n)$.
- Similarly, if μ is a Borel measure on $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} \frac{1}{(1 + |x|)^m}\, d\mu(x) < \infty$$

  for some $m \in \mathbb{N}$, then $\mu \in \mathscr{S}(\mathbb{R}^n)'$.
- Let $T \in \mathscr{S}(\mathbb{R}^n)'$ and $f \in \mathscr{C}^\infty(\mathbb{R}^n)$. We have already mentioned that the product fT is a distribution over $\mathbb{R}^n$. For fT to be tempered distribution, we need $f\psi$ to be a Schwarz function for any $\psi \in \mathscr{S}(\mathbb{R}^n)$. This will be the case, e.g., if f and all its derivatives have at most polynomial growth at infinity. If no restrictions on the growth of f and its derivatives are imposed, the product fT might not be a tempered distribution. Take, e.g., the tempered distribution $T \equiv 1$ and the smooth function $f(x) = e^x$.
- If $T \in \mathscr{S}(\mathbb{R}^n)'$ is a tempered distribution, then the derivative $\partial^\alpha T$ is also a tempered distribution for any $\alpha \in \mathbb{N}^n$.
- If $T \in \mathscr{S}(\mathbb{R}^n)'$, we define its Fourier transform $\mathscr{F}T$ by letting

$$\langle \mathscr{F}T, \psi \rangle = \langle\ T,\ \widehat{\psi}\ \rangle$$

  for any $\psi \in \mathscr{S}(\mathbb{R}^n)$. The map $\mathscr{F}T$ is a tempered distribution. Just as for smooth functions over $\mathbb{R}^n$, the equalities

$$\mathscr{F}(\partial^\alpha T) = (2i\pi\xi)^\alpha \mathscr{F}(T) \quad \text{and} \quad \partial^\alpha \mathscr{F}(T) = \mathscr{F}((-2i\pi x)^\alpha T)$$

  hold in $\mathscr{S}(\mathbb{R}^n)'$ for any $\alpha \in \mathbb{N}^n$.

Proposition 4.5 (Convolution). *Let $T \in \mathscr{S}(\mathbb{R}^n)'$ and $\psi \in \mathscr{S}(\mathbb{R}^n)$. We define the convolution $T * \psi$ by letting*

$$(T * \psi)(x) = \langle T, \psi(x - \cdot) \rangle$$

for any $x \in \mathbb{R}^n$. The map $T * \psi$ belongs to $\mathscr{C}^\infty(\mathbb{R}^n)$ and we have that

$$\partial^\alpha (T * \psi) = (\partial^\alpha T) * \psi = T * (\partial^\alpha \psi)$$

over $\mathbb{R}^n$ for any $\alpha \in \mathbb{N}^n$. Furthermore, $T * \psi$ has polynomial growth. In particular, $T * \psi$ is a tempered distribution over $\mathbb{R}^n$ and the equality

$$\mathscr{F}(T * \psi) = \mathscr{F}(T)\ \widehat{\psi}$$

holds in $\mathscr{S}(\mathbb{R}^n)'$.

4.2. Sobolev spaces

The main reference for this section is [7].

A function $u \in L^1_{\text{loc}}(U)$ has weak derivatives of order k in U if, for any $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq k$, the distributional derivative $\partial^\alpha u$ is actually a function and belongs to $L^1_{\text{loc}}(U)$, i.e. there exists $v_\alpha \in L^1_{\text{loc}}(U)$ such that

$$\int_U u(x) \partial^\alpha \varphi(x)\, dx = (-1)^{|\alpha|} \int_U v_\alpha(x) \varphi(x)\, dx$$

for any $\varphi \in \mathscr{C}_c^\infty(U)$. We write $v_\alpha = \partial^\alpha u$. When $u \in \mathscr{C}^k(U)$, then $\partial^\alpha u$ coincides with the usual partial derivative of u. In the sequel, when $u \in L^1_{\text{loc}}(U)$ is not assumed to have any particular regularity, $\partial^\alpha u$ will always stand for the distributional derivative.

For any integer $k \geq 1$, we define the Sobolev space

$$H^k(U) = \Big\{ u \in L^2(U) : \partial^\alpha u \in L^2(U),\ \forall \alpha \in \mathbb{N}^n, |\alpha| \leq k \Big\}.$$

Since $\mathscr{F}(\partial^\alpha u) = (2i\pi x)^\alpha \mathscr{F} u$ in $\mathscr{S}(\mathbb{R}^n)'$ and since a function belongs to $L^2(\mathbb{R}^n)$ if and only if its distributional Fourier transform does, the condition "$u \in L^2(\mathbb{R}^n)$ and $\partial^\alpha u \in L^2(\mathbb{R}^n)$" is equivalent to : $u \in L^2(\mathbb{R}^n)$ and $x^\alpha \mathscr{F} u \in L^2(\mathbb{R}^n)$. In particular, this allows one to give the following alternative definition of $H^k(\mathbb{R}^n)$:

$$H^k(\mathbb{R}^n) = \Big\{ u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1 + |\xi|^{2k}) |\mathscr{F} u(\xi)|^2\, dx < \infty \Big\}.$$

For any real $s > 0$, we finally define

$$\begin{aligned} H^s(\mathbb{R}^n) &= \Big\{ u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1 + |\xi|^{2s}) |\mathscr{F} u(\xi)|^2\, dx < \infty \Big\} \\ &= \Big\{ u \in L^2(\mathbb{R}^n) : (1 + |\xi|^2)^{s/2} \mathscr{F} u(\xi) \in L^2(\mathbb{R}^n) \Big\}. \end{aligned}$$

For any $s > 0$, the space $H^s(\mathbb{R}^n)$ is a Hilbert space, equipped with the inner product

$$(u, v)_{H^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (1 + |\xi|^2)^s \mathscr{F} u(\xi) \overline{\mathscr{F} v(\xi)}\, d\xi$$

Sobolev spaces are particularly approriate to study the regularity of distributional solutions u to the Laplace equation $-\Delta u = f$ under mild assumptions on u and f, described in the next definition.

Definition 4.6 (Weak Laplacian)**.** *Let $n \geq 1$ and $\Omega \subset \mathbb{R}^n$ be an open and bounded set. Let $g \in L^2(\Omega)$ and $u \in H^1(\Omega)$. We say that u satisfies $-\Delta u = f$ in the weak sense in Ω if*

$$\int_\Omega \langle \nabla u(x), \nabla \varphi(x) \rangle\ dx = \int_\Omega f(x) \varphi(x)\, dx$$

for any $\varphi \in \mathscr{C}_c^\infty(\Omega)$.

The following proposition is Theorem 2 of §6.3.1. in [7] and will be a key ingredient in our proofs.

Proposition 4.7 (Elliptic regularity, I)**.** *Let $n \geq 1$ and $\Omega \subset \mathbb{R}^n$ be an open and bounded set. Let $g \in H^k(\Omega)$ for some $k \in \mathbb{N}$ and $u \in H^1(\Omega)$ be such that $-\Delta u = g$ in the weak sense in Ω. Then $u \in H^{k+2}_{\mathrm{loc}}(\Omega)$, i.e $u \in H^{k+2}(V)$ for any open subset $V \subset \Omega$ such that $\overline{V} \subset \Omega$. In particular, u admits weak derivatives of order $k+2$ in Ω and we have that $-\Delta u = f$ almost everywhere in Ω, where $\Delta u = \sum_{i=1}^n \partial_i^2 u$ and $\partial_1^2 u, \ldots, \partial_n^2 u$ are weak derivatives of u.*

The following proposition is Corollary 2.17 in [9]. It will also play an important role in the proofs of this paper.

Proposition 4.8 (Elliptic regularity, II)**.** *Let $n \geq 1$ and B_1 be the unit open ball in $\mathbb{R}^n$. Let $f \in \mathscr{C}^{k,\alpha}(B_1)$ for some $\alpha \in (0,1)$ and $k \in \mathbb{N}$, and let $u \in H^1(B_1) \cap L^\infty(B_1)$ be such that $-\Delta u = f$ in the weak sense in B_1. Then $u \in \mathscr{C}^{k+2,\alpha}(B_1)$.*

We will use a direct generalization of this proposition, stated in the next corollary and proved in Appendix A

Corollary 4.9. *Let $n \geq 1$ and $\Omega \subset \mathbb{R}^n$ be an open subset. Let $f \in \mathscr{C}^{k,\alpha}_{\mathrm{loc}}(\Omega)$ for some $\alpha \in (0,1)$ and $k \in \mathbb{N}$, and let $u \in H^1_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ be such that $-\Delta u = f$ in the weak sense in Ω. Then $u \in \mathscr{C}^{k+2,\alpha}_{\mathrm{loc}}(\Omega)$.*

5. Introduction to fractional Laplacians

Different definitions of fractional Laplacians exist. Some rely on the Fourier transform, others on singular integrals or Sobolev spaces. These definitions all coincide for regular functions such as the Schwarz class but may differ in general, or at least have different domains of definiton. In this section, we provide a self-contained introduction based on the needs of the paper. The approach presented in this section uses the Fourier transform approach since it appears in this form in the proofs of our results.

The main references we used for this section are [28], [29], [17] and [22]. Let us fix $u \in \mathscr{S}(\mathbb{R}^n)$. Based on the fact that

$$\mathscr{F}((-\Delta)^\ell u) = (2\pi|\xi|)^{2\ell} \mathscr{F} u$$

for any integer $\ell \geq 0$, we naturally let

$$((-\Delta)^s u)(x) := (2\pi)^{2s} \mathscr{F}^{-1}(|\xi|^{2s} \mathscr{F} u(\xi))(x)$$

for any real $s > 0$ and any $x \in \mathbb{R}^n$. Let us make a comment about the factor $(2\pi)^{2s}$ appearing in the definition of $(-\Delta)^s u$. This is obivously a consequence of our choice of normalization in the definition of the Fourier transform since it derives from the fact that

$$\mathscr{F}((-\Delta)u) = (2\pi|\xi|)^2 \mathscr{F} u.$$

For another normalization in the Fourier transform, $\mathscr{F}_{a,b}$ say, defined by

$$(\mathscr{F}_{a,b} u)(\xi) := \frac{1}{b} \int_{\mathbb{R}^n} u(x) e^{-ia(x,\xi)}\, dx$$

for some $a > 0$ and $b > 0$, we have

$$\mathscr{F}_{a,b}^{-1} u(\xi) = b\Big(\frac{a}{2\pi}\Big)^n \int_{\mathbb{R}^n} u(x) e^{ia(x,\xi)}\, dx.$$

It is easy to show that

$$a^{2s} \mathscr{F}_{a,b}^{-1}(|\xi|^{2s} \mathscr{F}_{a,b} u(\xi))(x) = \mathscr{F}_{1,1}^{-1}(|\xi|^{2s} \mathscr{F}_{1,1} u(\xi))(x)$$

for all $x \in \mathbb{R}^n$. It follows that any choice of normalization in the Fourier transform leads to the same value of $(-\Delta)^s u$ if we let

$$((-\Delta)^s u)(x) = a^{2s} \mathscr{F}_{a,b}^{-1}(|\xi|^{2s} \mathscr{F}_{a,b} u(\xi))(x).$$

In this paper, we choose $a = 2\pi$ and $b = 1$.

When $s = n + \sigma$, with $n \in \mathbb{N}$ and $\sigma \in (0,1)$, taking the Fourier transform entails that

$$(-\Delta)^s u = (-\Delta)^\sigma ((-\Delta)^n u),$$

where $(-\Delta)^n$ is the usual differential operator $-\Delta$ taken n times. Let us therefore restrict to $s \in (0,1)$. It is proved in [22] (see Proposition 3.3) that, in this case, we have that

$$((-\Delta)^s u)(x) = c_{n,s} \lim_{\eta \to 0} \int_{\mathbb{R}^n \setminus B_\eta(x)} \frac{u(x) - u(z)}{|x-z|^{n+2s}} \, dz,$$

for any $x \in \mathbb{R}^n$ and some constant $c_{n,s}$ that only depends on n and s. Note that in the reference mentioned above, the normalization in the Fourier transform corresponds to $a = 1$ and $b = (2\pi)^{\frac{n}{2}}$ in our previous discussion. The value of the constant $c_{n,s}$ can be found in [29] (see Theorem 1) and is given by

$$c_{n,s} = \frac{s(1-s)4^s \Gamma(n/2 + s)}{|\Gamma(2-s)| \pi^{n/2}}.$$

Let us now try to extend the domain of $(-\Delta)^s$. It is easy to see that

$$\int_{\mathbb{R}^n} ((-\Delta)^s u)(x) v(x) \, dx = \int_{\mathbb{R}^n} u(x) ((-\Delta)^s v)(x) \, dx$$

for any $u, v \in \mathscr{S}(\mathbb{R}^n)$. Therefore, it is tempting to define the fractional Laplacian $(-\Delta)^s T$ of an arbitrary tempered distribution $T \in \mathscr{S}(\mathbb{R}^n)'$ (recall that Fourier transforms are involved in the definition of $(-\Delta)^s$) by letting

$$\langle (-\Delta)^s T, \psi \rangle := \langle T, (-\Delta)^s \psi \rangle \tag{2}$$

for any $\psi \in \mathscr{S}(\mathbb{R}^n)$. The issue in doing so is that $(-\Delta)^s \psi$ no longer belongs to $\mathscr{S}(\mathbb{R}^n)$ in general. The regularity of $(-\Delta)^s \psi$ is the content of the next proposition, which is stated in [28] but not proved. For the sake of completness, we provide a proof of this proposition in Appendix B.

Proposition 5.1. *Let $n \geq 1$ and $s \in (0,1)$. Let $u \in \mathscr{S}(\mathbb{R}^n)$. Then, $(-\Delta)^s u \in \mathscr{C}^\infty(\mathbb{R}^n)$ and we have that*

$$\sup_{x \in \mathbb{R}^n} |(1 + |x|^{n+2s}) \partial^\alpha ((-\Delta)^s u)(x)| < \infty \tag{3}$$

for any $\alpha \in \mathbb{N}^n$. In addition, for any $\alpha \in \mathbb{N}^n$ we have that

$$\begin{aligned} &\sup_{x \in \mathbb{R}^n} |(1 + |x|^{n+2s}) \partial^\alpha ((-\Delta)^s u)(x)| \\ &\qquad \lesssim |\partial^\alpha u|_{L^1(\mathbb{R}^n)} + \sup_{z \in \mathbb{R}^n} \Big((1 + |z|)^{n+2} |\nabla^2 (\partial^\alpha u)(z)| \Big), \end{aligned} \tag{4}$$

where, for any smooth function ψ we let $|\nabla^2 \psi(z)|$ stand for the operator norm of the Hessian matrix $\nabla^2 \psi(z)$ of ψ at z.

Therefore, let us define

$$\mathscr{S}_s(\mathbb{R}^n) := \{ \psi \in \mathscr{C}^\infty(\mathbb{R}^n) : \sup_{x \in \mathbb{R}^n} |(1 + |x|^{n+2s}) \partial^\alpha \psi(x)| < \infty, \ \forall \alpha \in \mathbb{N}^n \}.$$

For any $k \geq 1$, we similarly define $\mathscr{S}_s(\mathbb{R}^n, \mathbb{C}^k)$ as the collection of vector fields $\Psi = (\psi_1, \ldots, \psi_k)$ for which $\psi_i \in \mathscr{S}_s(\mathbb{R}^n)$ for any $i = 1, \ldots, k$. We endow $\mathscr{S}_s(\mathbb{R}^n)$ with the following convergence. A sequence $(\psi_k) \subset \mathscr{S}_s(\mathbb{R}^n)$ converges to $\psi \in \mathscr{S}_s(\mathbb{R}^n)$ in the space $\mathscr{S}_s(\mathbb{R}^n)$ if

$$\sup_{x \in \mathbb{R}^n} |(1 + |x|^{n+2s}) \partial^\alpha (\psi_k - \psi)(x)| \to 0$$

as $k \to \infty$ for any $\alpha \in \mathbb{N}^n$.

Definition 5.2 (Distributions for the fractional Laplacian)**.** *Let $s \in (0, 1)$. We let $\mathscr{S}_s(\mathbb{R}^n)'$ be the set of linear maps*

$$T : \mathscr{S}_s(\mathbb{R}^n) \to \mathbb{C}, \ \psi \mapsto \langle T, \psi \rangle$$

which are continuous with respect to the convergence on $\mathscr{S}_s(\mathbb{R}^n)$.

Similarly to tempered distributions, we let $\mathscr{S}_s^k(\mathbb{R}^n)'$ denote the space $(\mathscr{S}_s(\mathbb{R}^n)')^k$ for any $k \in \mathbb{N}$ with $k \geq 1$; see Definition 4.3 and the comments below.

Since the class of test functions for $\mathscr{S}_s(\mathbb{R}^n)'$ is richer than the one for $\mathscr{S}(\mathbb{R}^n)'$, i.e. $\mathscr{S}(\mathbb{R}^n) \subset \mathscr{S}_s(\mathbb{R}^n)$, we have that $\mathscr{S}_s'(\mathbb{R}^n) \subset \mathscr{S}(\mathbb{R}^n)$. Therefore, distributions in $\mathscr{S}_s(\mathbb{R}^n)'$ are more regular than those from $\mathscr{S}(\mathbb{R}^n)'$.

Proposition 5.1 entails that if a sequence $(\psi_k) \subset \mathscr{S}(\mathbb{R}^n)$ converges to 0 in the space $\mathscr{S}(\mathbb{R}^n)$, then the sequence $((-\Delta)^s \psi_k)$ converges to 0 in the space $\mathscr{S}_s(\mathbb{R}^n)$ as $k \to \infty$. In order to define the fractional Laplacian $(-\Delta)^s T$ of a tempered distribution T according to (2) and for $(-\Delta)^s T$ to be a tempered distribution as well, we will then require $T \in \mathscr{S}_s(\mathbb{R}^n)'$.

Definition 5.3 (Fractional Laplacian)**.** *Let $s \in (0, 1)$. For any $T \in \mathscr{S}_s(\mathbb{R}^n)'$, we let $(-\Delta)^s T$: $\mathscr{S}(\mathbb{R}^n) \to \mathbb{C}$ be the linear map defined by*

$$\langle (-\Delta)^s T, \psi \rangle := \langle T, (-\Delta)^s \psi \rangle$$

for any $\psi \in \mathscr{S}(\mathbb{R}^n)$. We have that $(-\Delta)^s T \in \mathscr{S}(\mathbb{R}^n)'$.

The class $\mathscr{S}_s(\mathbb{R}^n)$ is obviously closed with respect to derivation. However, it is not closed with respect to Fourier transform. Consequently, the space $\mathscr{S}_s(\mathbb{R}^n)'$ is closed with respect to derivation but not with respect to Fourier transform. In addition, it is easy to see that

$$\partial^\alpha (-\Delta)^s u = (-\Delta)^s \partial^\alpha u$$

for any $u \in \mathscr{S}(\mathbb{R}^n)$ and any $\alpha \in \mathbb{N}^n$. It immediately follows that

$$\partial^\alpha (-\Delta)^s T = (-\Delta)^s \partial^\alpha T$$

for any $T \in \mathscr{S}_s(\mathbb{R}^n)'$ and any $\alpha \in \mathbb{N}^n$.

We now need to understand which distributions of $\mathscr{S}(\mathbb{R}^n)'$ belong to $\mathscr{S}_s(\mathbb{R}^n)'$. It is trivial to see that every (signed) measure μ such that

$$\int_{\mathbb{R}^n} \frac{1}{1 + |x|^{n+2s}} \, d|\mu|(x) < \infty$$

belongs to $\mathscr{S}_s(\mathbb{R}^n)'$. In particular, every Borel probability measure over $\mathbb{R}^n$ belongs to $\mathscr{S}_s(\mathbb{R}^n)'$. This also entails that any function $u : \mathbb{R}^n \to \mathbb{C}$ belongs to $\mathscr{S}_s(\mathbb{R}^n)'$ provided that

$$\int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+2s}} \, dx < \infty.$$

In particular, we have that $L^p(\mathbb{R}^n) \subset \mathscr{S}_s(\mathbb{R}^n)'$ for any $p \in [1, +\infty]$.

We now provide a result that allows one to compute explicitly $(-\Delta)^s u$, which we prove in Appendix B.

Proposition 5.4. *Let $s \in (0,1)$ and $u \in \mathscr{S}_s(\mathbb{R}^n)'$. Then we have the following :*

1. *If $u \in H^{2s}(\mathbb{R}^n)$, then $(-\Delta)^s u \in L^2(\mathbb{R}^n)$ and we have that*
$$(-\Delta)^s u = (2\pi)^{2s} \mathscr{F}^{-1}(|\xi|^{2s} \mathscr{F} u)$$
in $L^2(\mathbb{R}^n)$;
2. *If $\mathscr{F} u \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $|\xi|^{2s} \mathscr{F} u(\xi) \in L^1(\mathbb{R}^n)$, then $(-\Delta)^s u \in \mathscr{C}_0(\mathbb{R}^n)$ and we have that*
$$((-\Delta)^s u)(x) = (2\pi)^{2s} \mathscr{F}^{-1}(|\xi|^{2s} \mathscr{F} u)(x)$$
for any $x \in \mathbb{R}^n$;
3. *If, for some open subset $\Omega \subset \mathbb{R}^n$ and some $\alpha \in (0, 2-2s)$, we have $u \in \mathscr{C}^{k,\beta}(\Omega)$ with $k = \lfloor 2s + \alpha \rfloor$ and $\beta = 2s + \alpha - k$, then $(-\Delta)^s u \in \mathscr{C}^0(\Omega)$ and we have that*
$$(-\Delta)^s(x) = c_{n,s} \lim_{\eta \to 0} \int_{\mathbb{R}^n \setminus B_\eta(x)} \frac{u(x) - u(z)}{|x - z|^{n+2s}} \, dz$$
for any $x \in \Omega$.

6. Recovering a probability measure from its spatial rank

Theorem 6.1. *Let $n \geq 1$. Let P be a Borel probability measure over $\mathbb{R}^n$ and assume that P admits a density $f_P \in \mathscr{S}(\mathbb{R}^n)$ with respect to the Lebesgue measure. Then $R_P \in \mathscr{C}^\infty(\mathbb{R}^n) \cap D(\mathscr{L}_n)$ and we have that*
$$f_P(x) = (\mathscr{L}_n R_P)(x)$$
for any $x \in \mathbb{R}^n$ (see Definition 2.3 for the definition of $\mathscr{L}_n$ and $D(\mathscr{L}_n)$).

Since $\mathscr{L}_1 = \frac{1}{2} \frac{d}{dx}$ and $R_P = 2F_P - 1$ over $\mathbb{R}$ when $n = 1$, where F_P is the usual cumulative distribution function of P, we recover the well-known fact that
$$F_P'(x) = f_P(x).$$

The proof of Theorem 6.1 requires the two following lemmas, which are proved in Appendix C.

Lemma 6.2. *Let $n \geq 2$ be an integer and $\alpha \in (0, n)$ be a real number. Then the Fourier transform of the tempered distribution $1/|x|^\alpha$ is given by*
$$\mathscr{F}\Big(\frac{1}{|x|^\alpha}\Big)(\xi) = \frac{\Gamma(\frac{n-\alpha}{2})}{\pi^{\frac{n}{2} - \alpha} \Gamma(\frac{\alpha}{2})} \frac{1}{|\xi|^{n-\alpha}}.$$

Lemma 6.3. *The derivative of the tempered distribution $1/|x|^{n-1}$ is given by*
$$\nabla\Big(\frac{1}{|x|^{n-1}}\Big) = -(n-1) \text{ P.V.}\Big(\frac{x}{|x|^{n+1}}\Big)$$
in $\mathscr{S}^n(\mathbb{R}^n)'$.

We are now able to give the proof of Theorem 6.1.

Proof of Theorem 6.1. Let us first recall that
$$R_P(x) = (K * f_P)(x)$$

for any $x \in \mathbb{R}^n$, where K is the kernel introduced in Section 2. Since $R_P \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$, we have that $R_P \in \mathscr{S}^n_{1/2}(\mathbb{R}^n)' \subset \mathscr{S}^n(\mathbb{R}^n)'$ (see Section 5) so that $R_P \in D(\mathscr{L}_n)$, the domain of $\mathscr{L}_n$, irrespective of n. Since $K \in \mathscr{S}^n(\mathbb{R}^n)'$ and $f_P \in \mathscr{S}(\mathbb{R}^n)$, Proposition 4.5 entails that $R_P \in \mathscr{C}^\infty(\mathbb{R}^n)$.

For $n = 1$, recall that $R_P = 2F_P - 1$, where

$$F_P(x) = \int_{-\infty}^{x} f_P(t)\, dt$$

is the cumulative distribution function of P. By the fundamental theorem of calculus, we then have

$$(\mathscr{L}_n R_P)(x) = \gamma_n\ (-\Delta)^{\frac{n-1}{2}} (\nabla \cdot R_P)(x) = \frac{1}{2} R_P'(x) = \frac{1}{2}(2F_P - 1)' = f_P(x).$$

Therefore, the claim is proved when $n = 1$.

Now assume that $n \geq 2$. Since $K \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$, we have that $K \in \mathscr{S}^n(\mathbb{R}^n)'$. It follows from Proposition 4.5 that

$$\mathscr{F}(R_P) = \mathscr{F}(K)\ \widehat{f_P} \tag{5}$$

in $\mathscr{S}^n(\mathbb{R}^n)'$, since $f_P \in \mathscr{S}(\mathbb{R}^n)$. Lemma 6.2 and the fact that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ yield

$$\mathscr{F}\Big(\frac{1}{|x|}\Big)(\xi) = \frac{\Gamma(\frac{n-1}{2})}{\pi^{\frac{n-1}{2}}} \frac{1}{|\xi|^{n-1}}$$

in $\mathscr{S}(\mathbb{R}^n)'$. From the identities stated in Section 4.1, we deduce that

$$\mathscr{F}(K) = \mathscr{F}\Big(\frac{x}{|x|}\Big) = -\frac{1}{2i\pi} \nabla \mathscr{F}\Big(\frac{1}{|x|}\Big) = -\frac{1}{2i\pi} \frac{\Gamma(\frac{n-1}{2})}{\pi^{\frac{n-1}{2}}} \nabla\Big(\frac{1}{|\xi|^{n-1}}\Big)$$

in $\mathscr{S}^n(\mathbb{R}^n)'$. Recalling that $x\Gamma(x) = \Gamma(x+1)$ for every $x > 0$, Lemma 6.3 yields

$$(\mathscr{F}K)(\xi) = \frac{\Gamma(\frac{n+1}{2})}{i\pi^{\frac{n+1}{2}}}\ \text{P.V.}\Big(\frac{\xi}{|\xi|^{n+1}}\Big) \tag{6}$$

in $\mathscr{S}^n(\mathbb{R}^n)'$. Equation (5) then rewrites

$$(\mathscr{F}R_P)(\xi) = \frac{\Gamma(\frac{n+1}{2})}{i\pi^{\frac{n+1}{2}}}\ \text{P.V.}\Big(\frac{\xi}{|\xi|^{n+1}}\Big) \widehat{f_P}(\xi)$$

in $\mathscr{S}^n(\mathbb{R}^n)'$. It is easy to see that

$$\Big(\xi,\ \text{P.V.}\big(\frac{\xi}{|\xi|^{n+1}}\big)\Big) := \sum_{i=1}^{n} \xi_i\ \text{P.V.}\big(\frac{\xi_i}{|\xi|^{n+1}}\big) = \frac{1}{|\xi|^{n-1}}$$

in $\mathscr{S}(\mathbb{R}^n)'$. Further note that

$$\big(\xi,\ (\mathscr{F}R_P)(\xi)\big) = \sum_{i=1}^{n} \xi_i \mathscr{F}((R_P)_i)(\xi) = \frac{1}{2i\pi} \sum_{i=1}^{n} \mathscr{F}(\partial_i (R_P)_i)(\xi) = \frac{1}{2i\pi} \mathscr{F}(\nabla \cdot R_P)(\xi)$$

in $\mathscr{S}(\mathbb{R}^n)'$. It follows that

$$\frac{1}{2i\pi} \mathscr{F}(\nabla \cdot R_P)(\xi) = \frac{\Gamma(\frac{n+1}{2})}{i\pi^{\frac{n+1}{2}}} \frac{1}{|\xi|^{n-1}} \widehat{f_P}(\xi) \tag{7}$$

in $\mathscr{S}(\mathbb{R}^n)'$. Let us now consider two cases. (A) Assume that $n \geq 3$ is odd. Therefore, $\frac{n-1}{2} \in \mathbb{N}$ and we have

$$\widehat{f_P}(\xi) = \frac{1}{2} \frac{\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n+1}{2})} |\xi|^{n-1} \mathscr{F}(\nabla \cdot R_P)(\xi) = \gamma_n\ \mathscr{F}((-\Delta)^{\frac{n-1}{2}} \nabla \cdot R_P)(\xi)$$

in $\mathscr{S}(\mathbb{R}^n)'$, where

$$\gamma_n = \frac{1}{2^n \pi^{\frac{n-1}{2}} \Gamma(\frac{n+1}{2})}.$$

Therefore, the equality

$$f_P = \gamma_n \ (-\Delta)^{\frac{n-1}{2}} (\nabla \cdot R_P) = \mathscr{L}_n(R_P)$$

holds in $\mathscr{S}(\mathbb{R}^n)'$. But $R_P \in \mathscr{C}^\infty(\mathbb{R}^n)$, which ensures that the r.h.s. of the last equality is a proper continuous function. Since f_P is also continuous and equality holds in the sense of distributions, equality also holds pointwise. (B) Assume that $n \geq 2$ is even. Since $n-2$ is even, we deduce from (7) that

$$\frac{\widehat{f_P}(\xi)}{|\xi|} = \frac{1}{2} \frac{\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n+1}{2})} |\xi|^{n-2} \mathscr{F}(\nabla \cdot R_P)(\xi) = \frac{1}{2} \frac{\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n+1}{2})} \frac{1}{(2\pi)^{n-2}} \mathscr{F}((-\Delta)^{\frac{n-2}{2}} (\nabla \cdot R_P))(\xi)$$

holds in $\mathscr{S}(\mathbb{R}^n)'$. Let us recall that $R_P \in \mathscr{S}^n_{1/2}(\mathbb{R}^n)'$. Since $\mathscr{S}^n_{1/2}(\mathbb{R}^n)'$ is closed with respect to derivation, we have that $u \in \mathscr{S}_{1/2}(\mathbb{R}^n)'$, with $u := (-\Delta)^{\frac{n-2}{2}} (\nabla \cdot R_P)$. It is clear that $\mathscr{F}u \in L^1_{\text{loc}}(\mathbb{R}^n)$ since $\widehat{f_P}(\xi)/|\xi| \in L^1_{\text{loc}}(\mathbb{R}^n)$ (recall that $n \geq 2$), and that $|\xi| \mathscr{F}u(\xi) \in L^1(\mathbb{R}^n)$ since $f_P \in \mathscr{S}(\mathbb{R}^n)$. It follows from Proposition 5.4 that $(-\Delta)^{\frac{1}{2}} (-\Delta)^{\frac{n-2}{2}} (\nabla \cdot R_P) \in \mathscr{C}_0(\mathbb{R}^n)$ and that

$$\widehat{f_P} = \gamma_n \ \mathscr{F}((-\Delta)^{\frac{1}{2}} (-\Delta)^{\frac{n-2}{2}} (\nabla \cdot R_P))$$

holds in $\mathscr{S}(\mathbb{R}^n)'$, where γ_n is the same constant as in (A). We deduce that

$$f_P = \gamma_n \ (-\Delta)^{\frac{1}{2}} (-\Delta)^{\frac{n-2}{2}} (\nabla \cdot R_P) = \mathscr{L}_n(R_P)$$

in $\mathscr{S}(\mathbb{R}^n)'$. Since both sides of this last equality are continuous, equality also holds pointwise over $\mathbb{R}^n$, which concludes the proof. ■

Definition 6.4 (Characteristic function). *Let $n \geq 1$ and P be a Borel probability measure over $\mathbb{R}^n$. For any $\xi \in \mathbb{R}^n$, let*

$$\phi_P(\xi) := \mathbb{E}[e^{-2i\pi\langle \xi, X \rangle}]$$

be the characteristic function of P, where X is a random n-vector with law P. Then, ϕ_P is the distributional Fourier transform of the tempered distribution P, i.e. $\phi_P = \mathscr{F}(P)$ in $\mathscr{S}(\mathbb{R}^n)'$ (see Section 4.1 for the definition and some properties of the space $\mathscr{S}(\mathbb{R}^n)'$ of tempered distributions). Letting $\varphi_P(\xi) = \mathbb{E}[e^{i\pi\langle \xi, X \rangle}]$ denote the usual characteristic function of P, we have that

$$\phi_P(\xi) = \varphi_P(-2\pi\xi)$$

for any $\xi \in \mathbb{R}^n$.

Theorem 6.5. *Let $n \geq 1$ and P be a Borel probability measure over $\mathbb{R}^n$. Then R_P belongs to the domain $D(\mathscr{L}_n)$ of $\mathscr{L}_n$. Furthermore, the distribution $\mathscr{F}(\mathscr{L}_n R_P)$ is a continous function over $\mathbb{R}^n$ and we have that*

$$\phi_P(\xi) = \mathscr{F}(\mathscr{L}_n R_P)(\xi)$$

for any $\xi \in \mathbb{R}^n$. In other words, the equality

$$P = \mathscr{L}_n(R_P)$$

holds in $\mathscr{S}(\mathbb{R}^n)'$.

The following lemma, which we need in order to prove Theorem 6.5, is stated in [2], Corollary 2.2.10.

Lemma 6.6. *Let Q and $(Q_k)_{k\geq 1}$ be Borel probability measures over $\mathbb{R}^n$ such that Q_k converges to Q in distribution as $k \to \infty$. Let $g : \mathbb{R}^n \to \mathbb{C}$ be a bounded and measurable map such that $Q(D_g) = 0$, where we let*

$$D_g := \{x \in \mathbb{R}^n : g \text{ is not continuous at } x\}.$$

Then $\int_{\mathbb{R}^n} g\, dQ_k \to \int_{\mathbb{R}^n} g\, dQ$ as $k \to \infty$.

Proof of Theorem 6.5. Assume first that there exists a sequence of probability measures (Q_k) over $\mathbb{R}^n$ such that (Q_k) converges in distribution (i.e. in law) to P as $k \to \infty$ and such that, for any k, Q_k admits a density $f_k \in \mathscr{S}(\mathbb{R}^n)$ with respect to the Lebesgue measure. For any k, let us denote R_{Q_k} the spatial rank associated to the probability measure Q_k. Since, for any k, Q_k admits the density $f_k \in \mathscr{S}(\mathbb{R}^n)$, Theorem 6.1 entails that

$$f_k(x) = (\mathscr{L}_n R_{Q_k})(x)$$

for any $x \in \mathbb{R}^n$ and any k. Let $\psi \in \mathscr{S}(\mathbb{R}^n)$. For any k, we then have that

$$\int_{\mathbb{R}^n} \psi(x) f_k(x)\, dx = \int_{\mathbb{R}^n} (R_{Q_k}, \mathscr{L}_n^*(\psi))\, dx. \tag{8}$$

We are going to show that the l.h.s. of (8) converges to $\int_{\mathbb{R}^n} \psi(x)\, dP(x)$ and that the r.h.s. of (8) converges to $\int_{\mathbb{R}^n} (R_P, \mathscr{L}_n^*(\psi))\, dx$ as $k \to \infty$.

In order to show that the l.h.s. of (8) converges, it is enough to observe that, since (Q_k) converges in distribution to P and ψ is continuous and bounded over $\mathbb{R}^n$, we have that

$$\int_{\mathbb{R}^n} \psi(x) f_k(x)\, dx = \int_{\mathbb{R}^n} \psi(x)\, dQ_k(x) \to \int_{\mathbb{R}^n} \psi(x)\, dP(x) \tag{9}$$

as $k \to \infty$. Let us now show that the r.h.s. of (8) converges. Let us start by showing that R_{Q_k} converges almost everywhere to R_P. For any $x \in \mathbb{R}^n$, let $g_x(z) := \frac{x-z}{\|x-z\|}\mathbb{I}[z \neq x]$ for any $z \in \mathbb{R}^n$. With the notations of Lemma 6.6, we have $D_{g_x} = \{x\}$. Let

$$A := \{x \in \mathbb{R}^n : P[\{x\}] > 0\}.$$

Then A is at most countable and we have $P[D_{g_x}] = 0$ for all $x \in \mathbb{R}^n \setminus A$. Since g_x is bounded and measurable for all $x \in \mathbb{R}^n$, Lemma 6.6 entails that

$$R_{Q_k}(x) = \int_{\mathbb{R}^n} g_x(z)\, dQ_k(z) \to \int_{\mathbb{R}^n} g_x(z)\, dP(z) = R_P(x)$$

for any $x \in \mathbb{R}^n \setminus A$ as $k \to \infty$. Since A is at most countable, we have that $R_{Q_k} \to R_P$ almost everywhere. In order to apply the dominated convergence theorem to the r.h.s. of (8), observe that $\mathscr{L}_n^*(\psi) \in L^1(\mathbb{R}^n)$. Indeed, if n is even, we have that $(-\Delta)^{\frac{n-2}{2}}\psi \in \mathscr{S}(\mathbb{R}^n)$ so that $(-\Delta)^{\frac{1}{2}}((-\Delta)^{\frac{n-2}{2}}\psi) \in \mathscr{S}_{1/2}(\mathbb{R}^n)$, whence

$$\mathscr{L}_n^*(\psi) = \nabla((-\Delta)^{\frac{1}{2}}(-\Delta)^{\frac{n-2}{2}}\psi) \in \mathscr{S}_{1/2}(\mathbb{R}^n, \mathbb{C}^n) \subset L^1(\mathbb{R}^n)$$

since $\psi \in \mathscr{S}(\mathbb{R}^n)$. If n is odd, then $\nabla\big((-\Delta)^{\frac{n-1}{2}}\psi\big)$ trivially belongs to $\mathscr{S}(\mathbb{R}^n, \mathbb{C}^n)$ which is a subset of $L^1(\mathbb{R}^n)$. Since the sequence of functions $(R_{Q_k})_k$ is uniformly norm-bounded by 1 and since $R_{Q_k} \to R_P$ almost everywhere as $k \to \infty$, Lebesgue's dominated convergence theorem entails that

$$\int_{\mathbb{R}^n} (R_{Q_k}, \mathscr{L}_n^*(\psi))\, dx \to \int_{\mathbb{R}^n} (R_{Q_P}, \mathscr{L}_n^*(\psi))\, dx \tag{10}$$

as $k \to \infty$. Putting (8), (9) and (10) together yields

$$\int_{\mathbb{R}^n} \psi(x)\, dP(x) = \int_{\mathbb{R}^n} (R_{Q_P}, \mathscr{L}_n^*(\psi))\, dx$$

for any $\psi \in \mathscr{S}(\mathbb{R}^n)$, which yields

$$P = \mathscr{L}_n(R_P)$$

in $\mathscr{S}(\mathbb{R}^n)'$.

It remains to show that there indeed exists a sequence of probability measures (Q_k) over $\mathbb{R}^n$ that converges in distribution to P and such that Q_k admits a density $f_k \in \mathscr{S}(\mathbb{R}^n)$ with respect to the Lebesgue measure. Let X be a random vector with law P. Let $\rho \in \mathscr{C}_c^\infty(\mathbb{R}^n)$ be such that $0 \leq \rho \leq 1$ and $\int_{\mathbb{R}^n} \rho(x)\,dx = 1$. In particular, ρ is a probability density over $\mathbb{R}^n$. Let then Y be a random vector with density ρ. For any k, let $X_k := X + \frac{1}{k}Y$. Since (X_k) converges to X in probability, we have that X_k converges to X in distribution. Observe that X_k admits the density $p_k := \rho_k * P$ with respect to the Lebesgue measure, where $\rho_k(x) := k^n\rho(kx)$ for any k and any $x \in \mathbb{R}^n$. In particular, $p_k \in \mathscr{C}^\infty(\mathbb{R}^n)$ since $\rho_k \in \mathscr{C}_c^\infty(\mathbb{R}^n)$. Since $X_k \to X$ in distribution, we have that

$$\int_{\mathbb{R}^n} g(x)p_k(x)\,dx \to \int_{\mathbb{R}^n} g(x)\,dP(x)$$

for any $g \in \mathscr{C}_b^0(\mathbb{R}^n)$ as $k \to \infty$. For any k, let $r_k > 0$ be such that

$$\int_{\mathbb{R}^n \setminus B_{r_k}} p_k(x)\,dx < \frac{1}{k}.$$

For any k, let $\chi_k \in \mathscr{C}_c^\infty(\mathbb{R}^n)$ be such that $0 \leq \chi_k \leq 1$ over $\mathbb{R}^n$, $\chi_k = 1$ over B_{r_k} and $\chi_k = 0$ over $\mathbb{R}^n \setminus B_{1+r_k}$. Let then $f_k(x) := \chi_k(x)p_k(x)$ for any k and any $x \in \mathbb{R}^n$. Since $(p_k) \subset \mathscr{C}^\infty(\mathbb{R}^n)$, we have that $(f_k) \subset \mathscr{C}_c^\infty(\mathbb{R}^n)$. In particular, $(f_k) \subset \mathscr{S}(\mathbb{R}^n)$. Let $g \in \mathscr{C}_b^0(\mathbb{R}^n)$. We have that

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^n} g(x)f_k(x)\,dx - \int_{\mathbb{R}^n} g(x)p_k(x)\,dx\Big| \\
&\qquad\leq \int_{\mathbb{R}^n} |g(x)|(\chi_k(x) - 1)p_k(x)\,dx \\
&\qquad\leq \int_{\mathbb{R}^n \setminus B_{r_k}} g(x)p_k(x)\,dx \\
&\qquad\leq \|g\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n \setminus B_{r_k}} p_k(x)\,dx \\
&\qquad\leq \frac{1}{k}\|g\|_{L^\infty(\mathbb{R}^n)}.
\end{aligned}$$

Since $\int_{\mathbb{R}^n} g(x)p_k(x)\,dx \to \int_{\mathbb{R}^n} g(x)\,dP(x)$, it follows that

$$\int_{\mathbb{R}^n} g(x)f_k(x)\,dx \to \int_{\mathbb{R}^n} g(x)\,dP(x)$$

for any $g \in \mathscr{C}_b^0(\mathbb{R}^n)$. Letting Q_k be the probability measure with density $f_k \in \mathscr{S}(\mathbb{R}^n)$ for any k yields the conclusion. ■

It is most natural to consider the particular situation where P admits a density f_P with respect to the Lebesgue measure over $\mathbb{R}^n$. In this case, Theorem 6.5 entails that the equality $f_P = \mathscr{L}_n R_P$ holds in the sense of tempered distributions. It is therefore natural to look for simple conditions that will allow one to compute $\mathscr{L}_n R_P$ explicitly, by succesively applying the differential operators involved in the definition of $\mathscr{L}_n$ to R_P, without restricting to smooth densities with rapid decay at infinity as in Theorem 6.1.

We start with the univariate case, which is already well-known.

Theorem 6.7 (Univariate case)**.** *Let P be a Borel probability measure over $\mathbb{R}$. Let $\Omega \subset \mathbb{R}$ be an open subset and assume that P is non-atomic over Ω. Then $R_P \in \mathscr{C}^0(\mathbb{R})$. If, in addition, we assume that P admits a density $f_\Omega \in L^1(\Omega)$ over Ω with respect to the Lebesgue measure, then R_P admits a weak derivative R'_P and we have that*

$$f_\Omega(x) = \gamma_1 R'_P(x) \tag{11}$$

for almost every $x \in \mathbb{R}$. If we further assume that f_Ω is continuous over Ω, then R_P is continuously differentiable over Ω and (11) holds pointwise on Ω, where R'_P now stands for the usual derivative.

In dimension $n \geq 2$, the operator $\mathscr{L}_n$ requires computing n derivatives when n is odd and $n-1$ derivatives when n is even. We therefore wish to reach a regularity of order at least $n-1$ in any case. We achieve this in the following proposition.

Proposition 6.8 (Intermediate regularity)**.** *Let $n \geq 2$ and P be a Borel probability measure over $\mathbb{R}^n$. Let $\Omega \subset \mathbb{R}^n$ be an open subset. We have the following :*

1. *If P is non-atomic over Ω, then $R_P \in \mathscr{C}^0(\Omega)$;*
2. *For some integer $\ell \in [1, n-1]$ and some real $p > \frac{n}{n-\ell}$, assume that P admits a density $f_\Omega \in L^1(\Omega)$ over Ω with respect to the Lebesgue measure and that $f_\Omega \in L^p_{\text{loc}}(\Omega)$. Let Z be a random n-vector with law P. Then $R_P(x) = \mathbb{E}[K(x-Z)] \in \mathscr{C}^\ell(\Omega)$ and we have that*

$$\partial^\alpha R_P(x) = \mathbb{E}[(\partial^\alpha K)(x-Z)]$$

   for any $x \in \Omega$ and any $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq \ell$;
3. *Under the assumptions of point 2 of this proposition, if we further assume that the density $f_{\mathbb{R}^n} =: f_P \in L^p(\mathbb{R}^n)$, then $\partial^\alpha R_P$ converges to 0 at infinity for any $\alpha \in \mathbb{N}^n$ such that $1 \leq |\alpha| \leq \ell$.*

In particular, if $f_\Omega \in L^{n+\varepsilon}_{\text{loc}}(\Omega)$ for some $\varepsilon > 0$ then $R_P \in \mathscr{C}^{n-1}(\Omega)$, and if $f_\Omega \in L^{n+\varepsilon}(\mathbb{R}^n)$ for some $\varepsilon > 0$ then $R_P \in \mathscr{C}^{n-1}_b(\mathbb{R}^n)$.

Proof of Proposition 6.8. 1. The fact that $R_P \in \mathscr{C}^0(\Omega)$ is a direct consequence of Lebesgue's dominated convergence theorem, provided P is non-atomic over Ω.

2. We are going to prove the result by induction. By the first part of the proof, we have that $R_P \in \mathscr{C}^0(\Omega)$. In addition, we trivially have that $|R_P(x)| \leq 1$ for any $x \in \mathbb{R}^n$, so that $R_P \in \mathscr{C}^0_b(\mathbb{R}^n)$.
Let $0 \leq k \leq \ell - 1$ and assume that $R_P \in \mathscr{C}^k(\Omega)$ with

$$\partial^\alpha R_P(x) = \mathbb{E}[(\partial^\alpha K)(x-X)] \tag{12}$$

for any $x \in \Omega$ and any $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq k$. Let us show that $R_P \in \mathscr{C}^{k+1}(\Omega)$ (and $R_P \in \mathscr{C}^{k+1}_b(\mathbb{R}^n)$ if $f_P \in L^p(\mathbb{R}^n)$) and that (12) holds for any $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq k+1$. To that purpose, let $\alpha \in \mathbb{N}^n$ with $|\alpha| = k$, $x \in \Omega$ and $r > 0$ be such that $\overline{B_r(x)} \subset \Omega$. Let $j \in \{1, \dots, n\}$ and e_j be the jth vector of the canonical basis of $\mathbb{R}^n$. We are going to show that

$$\frac{\partial^\alpha R_P(x+he_j) - \partial^\alpha R_P(x)}{h} \to \mathbb{E}[(\partial_j \partial^\alpha K)(x-Z)]$$

$h \to 0$ and that the limit is continuous over Ω. Without loss of generality, let us assume that $|h| < \kappa$ for some $\kappa < d(x, \partial\Omega)$, so that $x + he_j \in B_\kappa(x) \subset \Omega$. For any h, let

$$S_h := \{x + she_j : s \in [0,1]\}$$

be the line segment from x to $x + he_j$. Since $S_h \subset \Omega$ and P has a density over Ω, we have that

$$\frac{\partial^\alpha R_P(x+he_j) - \partial^\alpha R_P(x)}{h} = \mathbb{E}\left[\frac{(\partial^\alpha K)(x+he_j-Z) - (\partial^\alpha K)(x-Z)}{h}\mathbb{I}[Z \in \mathbb{R}^n \setminus S_h]\right]$$

for any h. In order to take the limit as $h \to 0$ under the above expectation, we will show that the integrand is a uniformly P-integrable family indexed by h and converges P-almost surely as $h \to 0$. Since $K \in \mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\})$, observe that

$$\frac{(\partial^\alpha K)(x + he_j - z) - (\partial^\alpha K)(x - z)}{h} = \int_0^1 (\partial_j \partial^\alpha K)(x + she_j - z)\, ds$$

for any $z \in \mathbb{R}^n \setminus S_h$. The latter obviously converges to $(\partial_j \partial^\alpha K)(x - z)$ as $h \to 0$, for any $z \in \mathbb{R}^n \setminus S_h$. Let us now show that the family of random vectors

$$\left(\int_0^1 (\partial_j \partial^\alpha K)(x + she_j - Z)\mathbb{I}[Z \in \mathbb{R}^n \setminus S_h]\, ds \right)_{|h|<\kappa}$$

is uniformly P-integrable. It is enough to show that there exists $\delta > 0$ such that

$$\sup_{|h|<\kappa} \mathbb{E}\Big[\Big| \int_0^1 (\partial_j \partial^\alpha K)(x + she_j - Z)\, ds \Big|^{1+\delta} \mathbb{I}[Z \in \mathbb{R}^n \setminus S_h] \Big] < \infty.$$

Let $\delta > 0$ be arbitrary for now and let us fix its value later on. Observe that $|\partial^\beta K(x)| \le C_\beta |x|^{-|\beta|}$ for any $x \in \mathbb{R}^n \setminus \{0\}$, any $\beta \in \mathbb{N}^n$ and some positive constant C_β. Therefore, there exists $C > 0$ such that

$$\begin{aligned} &\Big| \int_0^1 (\partial_j \partial^\alpha K)(x + she_j - z)\, ds \Big|^{1+\delta} \\ &\qquad \le \int_0^1 |(\partial_j \partial^\alpha K)(x + she_j - z)|^{1+\delta}\, ds \\ &\qquad \le C \int_0^1 \frac{1}{|x + she_j - z|^{(1+k)(1+\delta)}}\, ds \end{aligned}$$

for any $z \in \mathbb{R}^n \setminus S_h$ and any h, by Jensen's inequality. It follows from Fubini's theorem that

$$\begin{aligned} &\sup_{|h|<\kappa} \mathbb{E}\Big[\Big| \int_0^1 (\partial_j \partial^\alpha K)(x + she_j - Z)\, ds \Big|^{1+\delta} \mathbb{I}[Z \in \mathbb{R}^n \setminus S_h] \Big] \\ &\qquad \le C \sup_{|h|<\kappa} \int_0^1 \mathbb{E}\Big[\frac{1}{|x + she_j - Z|^{(1+k)(1+\delta)}} \mathbb{I}[Z \in \mathbb{R}^n \setminus S_h] \Big]\, ds \\ &\qquad \le C \sup_{|h|<\kappa} \sup_{s\in[0,1]} \mathbb{E}\Big[\frac{1}{|x + she_j - Z|^{(1+k)(1+\delta)}} \mathbb{I}[Z \in \mathbb{R}^n \setminus S_h] \Big]. \end{aligned}$$

Let us fix h such that $|h| < \kappa$ and $s \in [0, 1]$. We have that

$$\begin{aligned} &\mathbb{E}\Big[\frac{1}{|x + she_j - Z|^{(1+k)(1+\delta)}} \mathbb{I}[Z \in \mathbb{R}^n \setminus S_h] \Big] \\ &\qquad \le \frac{1}{r^{(1+k)(1+\delta)}} + \mathbb{E}\Big[\frac{1}{|x + she_j - Z|^{(1+k)(1+\delta)}} \mathbb{I}[Z \in B_r(x + she_j) \setminus S_h] \Big]. \end{aligned}$$

Since $B_r(x + she_j) \setminus S_h \subset \Omega$ and P admits a density f_Ω over Ω, Hölder's inequality yields

$$\begin{aligned} &\mathbb{E}\Big[\frac{1}{|x + she_j - Z|^{(1+k)(1+\delta)}} \mathbb{I}[Z \in B_r(x + she_j) \setminus S_h] \Big] \\ &\qquad = \int_{B_r(x+she_j)\setminus S_h} \frac{1}{|x + she_j - z|^{(1+k)(1+\delta)}} f_\Omega(z)\, dz \\ &\qquad \le \Big(\int_{B_r} \frac{1}{|z|^{q(1+k)(1+\delta)}}\, dz \Big)^{1/q} \Big(\int_{B_r(x+she_j)} |f_\Omega(z)|^p\, dz \Big)^{1/p}, \end{aligned}$$

where p is such that $f_\Omega \in L^p_{\mathrm{loc}}(\Omega)$ and $q = \frac{p}{p-1}$ is the conjugate exponent of p. The fact that $k \le \ell - 1$ and $p > \frac{n}{n-\ell}$ implies that $p > \frac{n}{n-(1+k)}$ and $q < \frac{n}{1+k}$. Let us therefore choose $\delta > 0$ small enough such that $q < \frac{n}{(1+k)(1+\delta)}$. In particular, we have that

$$\int_{B_r} \frac{1}{|z|^{q(1+k)(1+\delta)}}\, dz < \infty.$$

Since $\overline{B_\kappa(x)} \subset \Omega$ and $f_\Omega \in L^p_{\mathrm{loc}}(\Omega)$, we also have that

$$\int_{B_r(x+she_j)} |f_\Omega(z)|^p\, dz \le \int_{B_\kappa(x)} |f_\Omega(z)|^p\, dz < \infty$$

uniformly in $|h| < \kappa$ and $s \in [0,1]$. We deduce that

$$\sup_{|h|<\kappa} \mathbb{E}\Big[\Big| \int_0^1 (\partial_j \partial^\alpha K)(x + she_j - Z)\, ds\Big|^{1+\delta} \mathbb{I}[Z \in \mathbb{R}^n \setminus S_h]\Big] < \infty.$$

Therefore, the family of random vectors

$$\Big(\int_0^1 (\partial_j \partial^\alpha K)(x + she_j - Z)\mathbb{I}[Z \in \mathbb{R}^n \setminus S_h]\, ds \Big)_{|h|<\kappa}$$

is uniformly P-integrable. It follows from Lebesgue-Vitali's theorem that

$$\frac{\partial^\alpha R_P(x + he_j) - \partial^\alpha R_P(x)}{h} \to \mathbb{E}[(\partial_j \partial^\alpha K)(x - Z)]$$

for any $x \in \Omega$ as $h \to 0$. Let us show that $x \mapsto \mathbb{E}[(\partial_j \partial^\alpha K)(x - Z)]$ is continuous over Ω. Let $x \in \Omega$ and $(x_m) \subset \Omega$ be a sequence converging to x as $m \to \infty$. The familiy of random vectors $((\partial_j \partial^\alpha K)(x_m - Z))_{n\in\mathbb{N}}$ converges P-almost surely to $(\partial_j \partial^\alpha K)(x - Z)$ as $m \to \infty$ since P is non-atomic, and is uniformly P-integrable since

$$\sup_{m\in\mathbb{N}} \mathbb{E}[|(\partial_j \partial^\alpha K)(x_m - Z)|^{1+\eta}] \lesssim \sup_{m\in\mathbb{N}} \mathbb{E}\Big[\frac{1}{|x_m - Z|^{(1+k)(1+\eta)}}\Big] < \infty$$

for η small enough, by the previous computations. It follows that $\partial^\alpha R_P \in \mathscr{C}^1(\Omega)$ and that

$$\partial_j \partial^\alpha R_P(x) = \mathbb{E}[(\partial_j \partial^\alpha K)(x - Z)]$$

for any $x \in \mathbb{R}^n$. Since $\alpha \in \mathbb{N}^n$ with $|\alpha| = k$ was arbitrary, we deduce that $R_P \in \mathscr{C}^{k+1}(\Omega)$ and that (12) holds for any $\alpha \in \mathbb{N}^n$ with $|\alpha| \le k+1$. This proves the result, by induction.

3. The second part of the proof implies that $R_P \in \mathscr{C}^\ell(\mathbb{R}^n)$. Let $\alpha \in \mathbb{N}^n$ with $1 \le |\alpha| \le \ell$ and let us show that $\partial^\alpha R_P$ converges to 0 at infinity. For the sake of convenience, let us write $k := |\alpha|$. We have already noticed that

$$|\partial^\alpha R_P(x)| \le C\ \mathbb{E}\Big[\frac{1}{|x - Z|^k}\mathbb{I}[Z \ne x]\Big] =: C\ h(x)$$

for any $x \in \mathbb{R}^n$ and some positive constant C. Therefore, it is enough to show that h converges to at infinity. Let $(x_m) \subset \mathbb{R}^n$ be such that $|x_m| \to \infty$ as $m \to \infty$. A standard application of Lebesgue's dominated convergence theorem entails that

$$\mathbb{E}\Big[\frac{1}{|x_m - Z|^k}\mathbb{I}[Z \in \mathbb{R}^n \setminus B_1(x)]\Big] \to 0$$

as $m \to \infty$. Next observe that

$$\begin{aligned}
&\Big|\mathbb{E}\Big[\frac{1}{|x_m - Z|^k}\mathbb{I}[B_1(x)]\Big]\Big| \\
&\qquad = \int_{B_1(x_m)} \frac{1}{|z - x_m|^k} f_P(z)\, dz \\
&\qquad \le C\Big(\int_{B_1(x_m)} |f_P(z)|^p\, dz\Big)^{1/p}\Big(\int_{B_1} \frac{1}{|z|^{qk}}\, dz\Big)^{1/q},
\end{aligned}$$

where $q = \frac{p}{p-1}$ is the conjugate exponent of p. Since $p > \frac{n}{n-\ell}$ and $k \le \ell$, we have that $p > \frac{n}{n-k}$ whence $q < \frac{n}{k}$. In particular, we have that $qk < n$. It follows that

$$\int_{B_1} \frac{1}{|z|^{qk}}\, dz < \infty.$$

It remains to show that $\int_{B_1(x_m)} |f_P(z)|^p\, dz \to 0$ as $m \to \infty$. Since $f_P \in L^p(\mathbb{R}^n)$, let ν be the non-negative finite measure defined by

$$\nu(B) := \int_B |f_P(z)|^p\, dz$$

for any Borel subset $B \subset \mathbb{R}^n$. We then have that

$$\int_{B_1(x_m)} |f_P(z)|^p\, dz = \nu(B_1(x_m))$$

for any m. Furthermore, we have that $\nu(\mathbb{R}^n \setminus B_{|x_m|-1}) \to 0$ as $m \to \infty$ since ν is finite and $|x_m| \to \infty$ as $m \to \infty$. Consequently, we have that

$$\mathbb{E}\Big[\frac{1}{|x_m - Z|^k}\mathbb{I}[Z \in \mathbb{R}^n \setminus B_1(x)]\Big] \to 0$$

as $m \to \infty$. We deduce that $\partial^\alpha R_P$ converges to 0 at infinity for any $\alpha \in \mathbb{N}^n$ with $1 \le |\alpha| \le \ell$, which concludes the proof. ■

In order to reach the nth derivatives, one cannot take $\ell = n$ formally in Theorem 6.8. Indeed, if that were true and if $f_P \in L^\infty(\mathbb{R}^n)$, we would have $R_P \in \mathscr{C}^n(\mathbb{R}^n)$ which would imply that $f_P \in \mathscr{C}^0(\mathbb{R}^n)$ when n is odd by Theorem 6.5, which will not be the case in general.

When $f \in \mathscr{C}^0(\Omega)$, the next theorem however shows that we indeed have $R_P \in \mathscr{C}^n(\Omega)$ with $f_P = \mathscr{L}_n(R_P)$ under the additional very mild assumption that f_P belongs to some Hölder class.

Theorem 6.9 (Odd dimensions)**.** *Let $n \ge 3$ be odd and P be a Borel probability measure over $\mathbb{R}^n$. Let $\Omega \subset \mathbb{R}^n$ be an open subset and assume that P admits a density $f_\Omega \in L^1(\Omega)$ over Ω with respect to the Lebesgue measure. Then, we have the following :*

1. *If $f_\Omega \in L^p_{\mathrm{loc}}(\Omega)$ for some $p > n$, then $R_P \in \mathscr{C}^{n-1}(\Omega) \cap H^n_{\mathrm{loc}}(\Omega)$. In particular, R_P admits weak derivatives of order n in Ω and we have that*

$$f_\Omega(x) = (\mathscr{L}_n R_P)(x)$$

for almost any $x \in \Omega$.

2. *If $f_\Omega \in \mathscr{C}^{k,\alpha}_{\mathrm{loc}}(\Omega)$ for some $k \in \mathbb{N}$ and some $\alpha \in (0,1)$, then $R_P \in \mathscr{C}^{k+n}(\Omega)$ and we have that*

$$f_\Omega(x) = (\mathscr{L}_n R_P)(x)$$

for any $x \in \Omega$.

Proof of Theorem 6.9. 1. Let us recall that $\nabla g_P = R_P$ over Ω since P is non-atomic over Ω (see Definition 2.1 and the comments below). Let us also recall that

$$P = \gamma_n \ (-\Delta)^{\frac{n-1}{2}} R_P$$

in $\mathscr{S}(\mathbb{R}^n)'$ by Theorem 6.5. Since $\nabla \cdot R_P = \nabla \cdot (\nabla g_P) = \Delta g_P$ over Ω, we have that

$$-f_\Omega = \gamma_n (-\Delta)^{\frac{n+1}{2}} g_P$$

in $\mathscr{D}(\Omega)'$. Since $f_\Omega \in L^p_{\text{loc}}(\Omega)$ with $p > n$, Proposition 6.8 entails that $R_P \in \mathscr{C}^{n-1}(\Omega)$. In particular, we have that $g_P \in \mathscr{C}^n(\Omega)$.

We are now going to use elliptic regularity to improve on the regularity of g_P over Ω. Let us first fix an open and bounded subset $U \subset \Omega$ such that $\overline{U} \subset \Omega$. Let U_1 be an open and bounded subset such that $\overline{U} \subset U_1$ and $\overline{U_1} \subset \Omega$. Since $g_P \in \mathscr{C}^n(\Omega)$ and U_1 is bounded, we have that $(-\Delta)^{\frac{n-1}{2}} g_P \in H^1(U_1)$ with

$$- f_\Omega = \gamma_n \ (-\Delta)\big((-\Delta)^{\frac{n-1}{2}} g_P\big) \tag{13}$$

in $\mathscr{D}(U_1)'$. Since $f_\Omega \in L^p_{\text{loc}}(\Omega)$ with $p > n \geq 3$, we have that $f_\Omega \in L^2_{\text{loc}}(\Omega)$. In particular, $f_\Omega \in L^2(U_1)$ since U_1 is bounded. It follows from Proposition 4.7 that $(-\Delta)^{\frac{n-1}{2}} g_P \in H^2_{\text{loc}}(U_1)$ and that (13) holds almost everywhere. Let us fix another open and bounded subset U_2 such that $\overline{U} \subset U_2$ and $\overline{U_2} \subset U_1$. Since $g_P \in \mathscr{C}^{n-2}(\Omega)$ and U_2 is bounded, we have that $\Delta^{\frac{n-3}{2}} g_P \in H^1(U_2)$ satisfies

$$(-\Delta)\big((-\Delta)^{\frac{n-3}{2}} g_P\big) = (-\Delta)^{\frac{n-1}{2}} g_P.$$

Since $(-\Delta)^{\frac{n-1}{2}} g_P \in H^2(U_2)$, we have by elliptic regularity that $(-\Delta)^{\frac{n-3}{2}} g_P \in H^4_{\text{loc}}(U_2)$. Proceeding by induction, we construct open and bounded decreasing subsets

$$U_1 \supset U_2 \supset \ldots \supset U_{\frac{n+1}{2}} \supset U$$

such that $\overline{U} \subset U_k$ and $(-\Delta)^{\frac{n+1-2k}{2}} g_P \in H^{2k}_{\text{loc}}(U_k)$ for any $k = 1, \ldots \frac{n+1}{2}$. For $k = \frac{n+1}{2}$, we find that $g_P \in H^{n+1}_{\text{loc}}(U_{\frac{n+1}{2}})$. In particular, we have that $g_P \in H^{n+1}(U)$. Since U was arbitrary in the first place, we conclude that $g_P \in H^{n+1}_{\text{loc}}(\Omega)$. In particular, $R_P \in H^n_{\text{loc}}(\Omega)$.

2. The second part of the statement is proved similarly to the first part by replacing Proposition 4.7 by Proposition 4.8 and Corollary 4.9. Applying the same bootstrap method, we only need to prove that $(-\Delta)^{\frac{n+1-2k}{2}} g_P \in L^\infty_{\text{loc}}(\Omega)$ for any $k = 1, \ldots, \frac{n+1}{2}$ in order to apply Corollary 4.9. But this immediately follows from the fact that $g_P \in \mathscr{C}^n(\Omega)$ since $R_P \in \mathscr{C}^{n-1}(\Omega)$ and $R_P = \nabla g_P$ (see Definition 2.2). ■

Theorem 6.10 (Even dimensions)**.** *Let $n \geq 2$ be even and P be a Borel probability measure over $\mathbb{R}^n$. Assume that P admits a density $f_P \in L^1(\mathbb{R}^n)$ with respect to the Lebesgue measure. Further let $R_P^{(n-1)} := \gamma_n (-\Delta)^{\frac{n-2}{2}} (\nabla \cdot R_P)$. Then, we have the following :*

1. *If $n > 2$ and if $f_P \in L^p_{\text{loc}}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ for some $p > n$, then $R_P \in \mathscr{C}^{n-1}(\mathbb{R}^n) \cap H^n_{\text{loc}}(\mathbb{R}^n)$. In particular, R_P admits weak derivatives of order n in $\mathbb{R}^n$ and we have that*

$$f_P(x) = (\mathscr{L}_n R_P)(x) = ((-\Delta)^{\frac{1}{2}} R_P^{(n-1)})(x)$$

for almost any $x \in \mathbb{R}^n$. In addition, $R_P^{(n-1)} \in H^1(\mathbb{R}^n)$ and we have that

$$((-\Delta)^{\frac{1}{2}} R_P^{(n-1)})(x) = 2\pi \mathscr{F}^{-1}\Big(|\xi| \mathscr{F} R_P^{(n-1)}\Big)(x)$$

for almost any $x \in \mathbb{R}^n$.

2. If $f_P \in \mathscr{C}^{k,\alpha}(\mathbb{R}^n)$ for some $k \in \mathbb{N}$ and some $\alpha \in (0,1)$, then $R_P \in \mathscr{C}^{k+n}(\mathbb{R}^n)$ and we have that

$$f_P(x) = (\mathscr{L}_n R_P)(x)$$

for any $x \in \mathbb{R}^n$. In addition, we have that

$$(\mathscr{L}_n R_P)(x) = c_{n,1/2} \lim_{\eta\to 0} \int_{\mathbb{R}^n \setminus B_\eta(x)} \frac{R_P^{(n-1)}(x) - R_P^{(n-1)}(z)}{|x-z|^{n+1}}\, dz$$

for any $x \in \mathbb{R}^n$ (see Section 5 for the definition of $c_{n,1/2}$).

Proof Theorem 6.10. 1. Since $f_P \in L^p_{\text{loc}}(\mathbb{R}^n)$ with $p > n$, Proposition 6.8 entails that $R_P \in \mathscr{C}^{n-1}(\mathbb{R}^n)$. In particular $R_P^{(n-1)}$ is well-defined and continuous over $\mathbb{R}^n$. Let us show that $R_P^{(n-1)} \in L^2(\mathbb{R}^n)$. Proposition 6.8 entails that

$$R_P^{(n-1)}(x) = \gamma_n \mathbb{E}[((-\Delta)^{\frac{n-2}{2}}(\nabla \cdot K))(x-Z)],$$

where Z is a random n-vector with law P. It follows that

$$|R_P^{(n-1)}(x)| \lesssim \mathbb{E}\Big[\frac{1}{|x-Z|^{n-1}}\Big] =: h(x).$$

Therefore, it is enough to show that $h \in L^2(\mathbb{R}^n)$. Let us write

$$h(x) = \Big(\frac{1}{|z|^{n-1}} * f_P\Big)(x) = (u_1 * f_P)(x) + (u_2 * f_P)(x),$$

with $u_1(z) = \frac{1}{|z|^{n-1}}\mathbb{I}[|z| < 1]$ and $u_2(z) = \frac{1}{|z|^{n-1}}\mathbb{I}[|z| > 1]$. Since $u_1 \in L^1(\mathbb{R}^n)$ and $f_P \in L^2(\mathbb{R}^n)$, Hausdorff-Young's inequality entails that $u_1 * f_P \in L^2(\mathbb{R}^n)$. Since $n > 2$, we have that $u_2 \in L^2(\mathbb{R}^n)$. Since $f_P \in L^1(\mathbb{R}^n)$, Hausdorff-Young's inequality yields $u_2 * f_P \in L^2(\mathbb{R}^n)$. It follows that $h \in L^2(\mathbb{R}^n)$, hence $R_P^{(n-1)} \in L^2(\mathbb{R}^n)$. Recall that $f_P = (-\Delta)^{\frac{1}{2}} R_P^{(n-1)}$ in $\mathscr{S}(\mathbb{R}^n)'$, with $R_P^{(n-1)} \in L^2(\mathbb{R}^n)$ and $f_P \in L^2(\mathbb{R}^n)$. Arguing as in the proof of Proposition 5.4, it is easy to show that this implies that $R_P^{(n-1)} \in H^1(\mathbb{R}^n)$. Proposition 5.4 therefore entails that

$$(-\Delta)^{\frac{1}{2}} R_P^{(n-1)} = 2\pi \mathscr{F}^{n-1}\big(|\xi| \mathscr{F} R_P^{(n-1)}\big)$$

in $L^2(\mathbb{R}^n)$ and that the equality

$$f_P = (-\Delta)^{\frac{1}{2}} R_P^{(n-1)}$$

also holds in $L^2(\mathbb{R}^n)$, i.e. almost everywhere. The same bootstrap argument than in the proof of Theorem 6.9 yields $g_P \in H^{n+1}_{\text{loc}}(\mathbb{R}^n)$, whence $R_P \in H^n_{\text{loc}}(\mathbb{R}^n)$.

2. The proof proceeds in two main steps. We first show that the fact that $f_P \in \mathscr{C}^{k,\alpha}(\mathbb{R}^n)$ entails that $R_P^{(n-1)} \in \mathscr{C}^{k+1,\alpha}_{\text{loc}}(\mathbb{R}^n)$. We will then show that this implies that $R_P \in \mathscr{C}^{k+n}(\mathbb{R}^n)$. Irrespective of the value of $k \in \mathbb{N}$, we will have that $R_P^{(n-1)} \in \mathscr{C}^{1,\alpha}_{\text{loc}}(\mathbb{R}^n)$, which will entail that

$$(-\Delta)^{1/2} R_P^{(n-1)}(x) = c_{n,1/2} \lim_{\eta\to 0} \int_{\mathbb{R}^n \setminus B_\eta(x)} \frac{R_P^{(n-1)}(x) - R_P^{(n-1)}(z)}{|x-z|^{n+1}}\, dz$$

for any $x \in \mathbb{R}^n$, by Proposition 5.4.

Let us first show that $R_P^{(n-1)} \in \mathscr{C}^{k+1}(\mathbb{R}^n)$. Observe that $f_P \in L^\infty$, since $f_P \in \mathscr{C}^{0,\alpha}(\mathbb{R}^n)$. Theorem 6.8 then entails that $R_P \in \mathscr{C}^{n-1}_b(\mathbb{R}^n)$. In particular, $R_P^{(n-1)} \in \mathscr{C}^0(\mathbb{R}^n)$ is bounded over $\mathbb{R}^n$ so that $(-\Delta)^{1/2} R_P^{(n-1)}$ is a well-defined tempered distribution. Recall that the equality

$$f_P = (-\Delta)^{1/2} R_P^{(n-1)}$$

holds in $\mathscr{S}(\mathbb{R}^n)'$ by Theorem 6.5. If $k=0$, we have that $f_P \in \mathscr{C}^{0,\alpha}(\mathbb{R}^n)$ and $R_P^{(n-1)} \in L^\infty(\mathbb{R}^n)$ with $f_P = (-\Delta)^{1/2} R_P^{(n-1)}$ in $\mathscr{S}(\mathbb{R}^n)'$. Proposition 2.8 in [28] then entails that $R_P^{(n-1)} \in \mathscr{C}^{1,\alpha}(\mathbb{R}^n)$. Now assume that $k \geq 1$. Since $f_P \in L^1(\mathbb{R}^n) \subset \mathscr{S}_{1/2}(\mathbb{R}^n)'$, then $(-\Delta)^{1/2} f_P$ is well-defined. Furthermore, we have that $(-\Delta)^{1/2} f_P \in \mathscr{C}^{k-1,\alpha}(\mathbb{R}^n)$ by Proposition 2.7 in [28], since $f_P \in \mathscr{C}^{k,\alpha}(\mathbb{R}^n)$. We then have that

$$(-\Delta)^{1/2} f_P = -\Delta R_P^{(n-1)}$$

in $\mathscr{S}(\mathbb{R}^n)'$ with $(-\Delta)^{1/2} f_P \in \mathscr{C}^{k-1,\alpha}(\mathbb{R}^n)$ and $R_P^{(n-1)} \in H^1_{\text{loc}}(\mathbb{R}^n) \cap L^\infty_{\text{loc}}(\mathbb{R}^n)$ (recall that we just showed that $R_P^{(n-1)} \in \mathscr{C}^{1,\alpha}(\mathbb{R}^n)$ when considering the case $k=0$). Therefore, Corollary 4.9 entails that $R_P^{(n-1)} \in \mathscr{C}^{k+1,\alpha}_{\text{loc}}(\mathbb{R}^n)$.

Let us now show that $R_P \in \mathscr{C}^{k+n}(\mathbb{R}^n)$. Recall that $R_P = \nabla g_P$ (see Definition 2.2 and the comments below) and that $R_P \in \mathscr{C}^{n-1}(\mathbb{R}^n)$, so that $g_P \in \mathscr{C}^n(\mathbb{R}^n)$. Consequently, we have that

$$-R_P^{(n-1)}(x) = \gamma_n (-\Delta)^n g_P(x)$$

for any $x \in \mathbb{R}^n$. Since $R_P^{(n-1)} \in \mathscr{C}^{k+1,\alpha}_{\text{loc}}(\mathbb{R}^n)$ and $(-\Delta)^{\frac{n-2}{2}} g_P \in H^1_{\text{loc}}(\mathbb{R}^n) \cap L^\infty_{\text{loc}}(\mathbb{R}^n)$, we have that $(-\Delta)^{\frac{n-2}{2}} g_P \in \mathscr{C}^{k+3,\alpha}_{\text{loc}}(\mathbb{R}^n)$ by Corollary 4.9. Repeating the argument recursively, we find that $g_P \in \mathscr{C}^{k+n+1,\alpha}_{\text{loc}}(\mathbb{R}^n)$. In particular, we have that $R_P \in \mathscr{C}^{k+n,\alpha}_{\text{loc}}(\mathbb{R}^n)$, which entails that $R_P \in \mathscr{C}^{k+n}(\mathbb{R}^n)$. This concludes the proof. ■

7. Depth regions and probability content

Let us consider a probability measure P over $\mathbb{R}^n$, with $n \geq 2$. For any $\beta \in [0,1)$ and $u \in S^{n-1}$, let us recall that a spatial quantile of order β in direction u for P is an arbitrary minimizer of the objective function $O^P_{\beta,u}$ introduced in Section 1. When P is not supported on a single line of $\mathbb{R}^n$, Theorem 1 in [25] entails that the spatial quantile of order β in direction u for P is unique for any $\beta \in [0,1)$ and $u \in S^{n-1}$; we denote it by $Q_P(\beta u)$.

Definition 7.1 (Depth contours and regions). *Let $n \geq 2$ and P be a probability measure over $\mathbb{R}^n$. Assume that P is not supported on a single line of $\mathbb{R}^n$. For any $\beta \in [0,1)$, let us recall that the depth region $\mathcal{D}^\beta_P$ and the depth contour $\mathcal{C}^\beta_P$ of P of order β are defined as*

$$\mathcal{D}^\beta_P = \Big\{ Q_P(\alpha u) : \alpha \in [0,\beta],\ u \in S^{n-1} \Big\}$$

and

$$\mathcal{C}^\beta_P = \Big\{ Q_P(\beta u) : u \in S^{n-1} \Big\}.$$

When P is non-atomic and not supported on a line of $\mathbb{R}^n$, Proposition 6.1 in [15] entails that the map Q_P is continuous over the open unit ball B_1. It directly follows that

$$\mathcal{D}^\beta_P = Q_P(\beta\, \overline{B_1})$$

is compact and arc-connected and that

$$\mathcal{C}^\beta_P = Q_P(\beta\, S^{n-1})$$

is compact and arc-connected as well. Furthermore, the depth regions $(D^\beta_P)_{\beta\in[0,1)}$ are obivously nested while depth-contours $(\mathcal{C}^\beta_P)_{\beta\in[0,1)}$ are obviously disjoint. Although quantile regions are convex in most cases, they may fail to be convex in general ; see [19].

In order to derive regularity properties of depth contours, let us first rewrite depth contours in terms of the rank function R_P. Theorem 6.1 in [15] entails that $x = Q_P(\alpha u)$ if and only if $R_P(x) = \alpha u$. This allows one to rewrite

$$\mathcal{D}_P^\beta = \Big\{x \in \mathbb{R}^n : |R_P(x)| \le \beta\Big\}$$

and

$$\mathcal{C}_P^\beta = \Big\{x \in \mathbb{R}^n : |R_P(x)| = \beta\Big\}.$$

The results of Section 6 may now easily be used to derive regularity properties of depth contours, as we show in the next proposition.

Proposition 7.2 (Regularity of depth contours)**.** *Let $n \ge 2$ and P be a probability measure over $\mathbb{R}^n$. Assume that P admits a density $f_P \in L^1(\mathbb{R}^n)$ with respect to the Lebesgue measure. We have the following :*

1. *If $f_P \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ for some real $p > \frac{n}{n-\ell}$ and some integer $\ell \in [1, n-1]$, then the depth contour $\mathcal{C}_P^\beta$ is an $(n-1)$-dimensional manifold of class $\mathscr{C}^\ell$ for any $\beta \in [0,1)$;*
2. *If $f_P \in \mathscr{C}^{k,\alpha}(\mathbb{R}^n)$ for some $k \in \mathbb{N}$ and some $\alpha \in (0,1)$, then the depth contour $\mathcal{C}_P^\beta$ is an $(n-1)$-dimensional manifold of class $\mathscr{C}^{k+n}$ for any $\beta \in [0,1)$.*

Proof of Proposition 7.2. Proposition 6.8, Theorem 6.9 and Theorem 6.10 yield that R_P has the stated regularity, $R_P \in \mathscr{C}^j(\mathbb{R}^n)$ say, and that

$$\partial^\alpha R_P(x) = \mathbb{E}[(\partial^\alpha K)(x - Z)]$$

for any $x \in \mathbb{R}^n$ and any $\alpha \in \mathbb{N}^n$ with $|\alpha| \le j$, where Z is a random n-vector with law P. Let us fix $\beta \in [0,1)$. Let $g_\beta(x) := |R_P(x)|^2 - \beta^2$. Then $g_\beta \in \mathscr{C}^j(\mathbb{R}^n)$ since the map $z \mapsto |z|^2$ is smooth over $\mathbb{R}^n$. We obviously have that

$$\mathcal{C}_P^\beta = \{x \in \mathbb{R}^n : g_\beta(x) = 0\}.$$

Let $z = (z', z_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ be such that $z \in \mathcal{C}_\beta$ and let us assume that $\nabla g_\beta(z) \neq 0$. The implicit function theorem entails that there exists an open neighbourhood $U \subset \mathbb{R}^{n-1}$ of z', an open neighbourhood $V \subset \mathbb{R}^n$ of z and a map $\varphi \in \mathscr{C}^j(U, \mathbb{R})$ such that $\varphi(z') = z_n$ and such that

$$V \cap \mathcal{C}_\beta = \{(x', \varphi(x')) : x' \in U\}.$$

In other words, in a neighbourhood of z, $\mathcal{C}_\beta$ is the graph of a function of class $\mathscr{C}^j$, which proves the claim.

It remains to show that $\nabla g_\beta(z) \neq 0$. Since $R_P \in \mathscr{C}^1(\mathbb{R}^n)$, we have that

$$\nabla g_\beta(z) = 2 J_{R_P}(z)^T R_P(z),$$

where $J_{R_P}(z)^T$ stands for the transpose of the Jacobian matrix of R_P at z. Recall that $\partial_j R_P(z) = \mathbb{E}[(\partial_j K)(z - Z)]$ and that

$$J_K(x) = \frac{1}{|x|}\Big(I_n - \frac{xx^T}{|x|^2}\Big)$$

for any $x \in \mathbb{R}^n \setminus \{0\}$, where I_n stands for the $n \times n$ identity matrix. Therefore, we have that

$$J_{R_P}(z) = \mathbb{E}\bigg[\frac{1}{|z - Z|}\Big(I_n - \frac{(z-Z)(z-Z)^T}{|z-Z|^2}\Big)\mathbb{I}[Z \neq z]\bigg].$$

The matrix J_{R_P} is obviously symmetric and non-negative definite. Let us show that it is positive definite. Assume, ad absurdum, that there exists $v \in \mathbb{R}^n$ such that $|v| = 1$ and $v^T J_{R_P}(z) v = 0$, i.e.

$$\mathbb{E}\bigg[\frac{1}{|z-Z|}\Big(1 - \Big(v, \frac{z-Z}{|z-Z|}\Big)^2\Big)\mathbb{I}[Z \neq z]\bigg] = 0.$$

We then have that
$$\frac{1}{|z-Z|}\Big(1-\Big(v,\frac{z-Z}{|z-Z|}\Big)^2\Big)\mathbb{I}[Z\neq z]=0$$
P-almost surely. Since P admits a density, we have that $\frac{1}{|z-Z|}\mathbb{I}[Z\neq z]\neq 0$ with P-probability 1. Consequently, we have that
$$\Big|\Big(v,\frac{z-Z}{|z-Z|}\Big)\Big|=1$$
with P-probability 1. This implies that P is supported on the line through z with direction v, a contradiction. We deduce that $J_{R_P}(z)$ is positive definite, hence invertible. It follows that $J_{R_P}(z)^T R_P(z)\neq 0$, whence $\nabla g_\beta(z)\neq 0$. This concludes the proof. ∎

Unlike center-outward quantiles based on optimal transport [12], spatial quantile regions are not indexed by their probability content, i.e. we do not have $P[\mathcal{D}_P^\beta]=\beta$ in general. However, one can in principle re-index quantile regions so that they match their probability content. Assume that P admits a density f_P over $\mathbb{R}^n$ such that $f_P(x)>0$ for any $x\in\mathbb{R}^n$. Let $\theta_P(\beta)=P[\mathcal{D}_P^\beta]$ for any $\beta\in[0,1)$. Since quantile regions are nested, the map θ_P is monotone non-decreasing. The assumptions on P further ensure that $\theta_P:[0,1)\to[0,1)$ is continuous, strictly increasing and surjective. In particular, θ_P is bijective. Therefore, the re-indexed quantile regions
$$\widetilde{\mathcal{D}}_P^\beta:=\mathcal{D}_P^{\theta_P^{-1}(\beta)}$$
match their probability content, i.e. we have $P[\tilde{\mathcal{D}}_P^\beta]=\beta$ for any $\beta\in[0,1)$. We similarly define the re-indexed quantile contours
$$\widetilde{\mathcal{C}}_P^\beta:=\mathcal{C}_P^{\theta_P^{-1}(\beta)}$$
for any $\beta\in[0,1)$. This suggests defining an alternative rank function $\widetilde{R}_P(x)$. To do so, observe that $x\in\widetilde{\mathcal{D}}_P^\beta$ if and only if
$$\Big|\frac{\beta}{\theta_P^{-1}(\beta)}R_P(x)\Big|=\beta.$$
When the previous equality holds, we have $\beta=\theta(|R_P(x)|)$. This suggests letting
$$\widetilde{R}_P(x)=\theta_P(|R_P(x)|)\frac{R_P(x)}{|R_P(x)|}$$
for any $x\in\mathbb{R}^n$. It follows that
$$\widetilde{\mathcal{D}}_P^\beta=\{x\in\mathbb{R}^n:|\widetilde{R}_P(x)|\leq\beta\}$$
and
$$\widetilde{\mathcal{C}}_P^\beta=\{x\in\mathbb{R}^n:|\widetilde{R}_P(x)|=\beta\}$$
for any $\beta\in[0,1)$. Letting Z denote a random n-vector with law P, it is clear that $\theta_P(|R_P(Z)|)$ is uniformly distributed over $[0,1)$. Indeed, we have
$$P\Big[\theta_P(|R_P(Z)|)\leq\beta\Big]=P\Big[|R_P(Z)|\leq\theta_P^{-1}(\beta)\Big]=P\big[\widetilde{\mathcal{D}}_P^\beta\big]=\beta$$
for any $\beta\in[0,1)$. Actually, it is easy to see that θ is the cumulative distribution function of $|R_P(Z)|$.

By Theorem 6.9 and Theorem 6.10, we have that
$$f_P(x)=\gamma_n(-\Delta)^{\frac{n-1}{2}}(\nabla\cdot R_P)(x)$$
for any $x\in\mathbb{R}^n$. Interchanging the order of the differential operators, we have
$$f_P=\gamma_n\nabla\cdot\big((-\Delta)^{\frac{n-1}{2}}R_P\big)$$

over $\mathbb{R}^n$, where $(-\Delta)^{\frac{n-1}{2}} R_P$ stands for the operator $(-\Delta)^{\frac{n-1}{2}}$ applied componentwise to R_P. For any (regular) open and bounded subset $\Omega \subset \mathbb{R}^n$, the divergence theorem then yields

$$P[\Omega] = \int_\Omega f_P(x)\, dx = \gamma_n \int_{\partial\Omega} \Big((-\Delta)^{\frac{n-1}{2}} R_P(x), \nu(x) \Big)\, dH_{n-1}(x), \tag{14}$$

for any $\beta \in [0,1)$, where H_{n-1} stands for the $(n-1)$-dimensional Hausdorff measure and $\nu(x)$ is the outer unit normal vector to Ω at x. The probability content of an arbitrary open subset is then controlled by $(-\Delta)^{\frac{n-1}{2}} R_P$. Notice that $(-\Delta)^{\frac{n-1}{2}} R_P$ and R_P actually coincide when $n = 1$. Therefore, Equation (14) is a multivariate analog of the well-known equality

$$P[Z \in [a,b]] = F_P(b) - F_P(a)$$

when $n = 1$ and where F_P stands for the usual cumulative distribution function.

8. Localizibility issues

In this section we investigate the local properties of the operator $\mathscr{L}_n$. As we have already noticed, the operator $\mathscr{L}_n = (-\Delta)^{\frac{n-1}{2}} \nabla\cdot$ displays a substantially different behaviour in odd and even dimensions. This is due the nature of $(-\Delta)^{\frac{n-1}{2}}$, which depends on whether $\frac{n-1}{2}$ is an integer or not. When $\frac{n-1}{2} \in \mathbb{N}$, then $(-\Delta)^{\frac{n-1}{2}}$ is the classical differential operator that consists in applying the Laplacian $-\Delta$ succesively $\frac{n-1}{2}$ times. This operator is local in nature : if f_1 and f_2 are smooth functions that coincide over an open subset $U \subset \mathbb{R}^n$, then $(-\Delta)^{\frac{n-1}{2}} f_1$ and $(-\Delta)^{\frac{n-1}{2}} f_2$ also coincide over U. When n is even, we have that $\frac{n-1}{2} \in \mathbb{R} \setminus \mathbb{N}$. We then write $(-\Delta)^{\frac{n-1}{2}} = (-\Delta)^{1/2} (-\Delta)^{\frac{n-2}{2}}$. Despite the fact that $(-\Delta)^{1/2}$ acts like a derivation in terms of regularity (see Proposition 2.6 in [28]), it is also known to be a non-local operator.

It is already well-known that multivariate spatial ranks characterize probability measures in arbitrary dimension n : if P and Q are Borel probability measures over $\mathbb{R}^n$ and if $R_P(x) = R_Q(x)$ for any $x \in \mathbb{R}^n$, then $P = Q$ (see Theorem 2.5 in [14]). When n is odd, we provide a refinement of this result in the next proposition, thanks to the local nature of $\mathscr{L}_n$.

Proposition 8.1. *Let $n \geq 1$ be odd. Let also P and Q be a Borel probability measures over $\mathbb{R}^n$. Let $\Omega \subset \mathbb{R}^n$ be an open subset and assume that $R_P(x) = R_Q(x)$ for any $x \in \Omega$. Then, P and Q coincide over Ω, i.e. $P(E) = Q(E)$ for any Borel subset $E \subset \Omega$.*

Proof of Proposition 8.1. By Theorem 6.5, we have that

$$\int_{\mathbb{R}^n} \psi(x)\, dP(x) = \int_{\mathbb{R}^n} (R_P(x), (\mathscr{L}_n^* \psi)(x))\, dx$$

and

$$\int_{\mathbb{R}^n} \psi(x)\, dQ(x) = \int_{\mathbb{R}^n} (R_Q(x), (\mathscr{L}_n^* \psi)(x))\, dx$$

for any $\psi \in \mathscr{S}(\mathbb{R}^n)$. In particular, the above equalities hold for any $\psi \in \mathscr{C}_c^\infty(\Omega)$. Let us observe that $\mathscr{L}_n^* = \gamma_n \nabla(-\Delta^{\frac{n-1}{2}})$ is a (non-fractional) differential operator, since $\frac{n-1}{2}$ is an integer. In particular, $\mathscr{L}_n^* \psi$ is also supported in Ω. Since $R_P = R_Q$ over Ω, we then have that

$$\int_\Omega \psi(x)\, dP(x) = \int_\Omega \psi(x)\, dQ(x)$$

for any $\psi \in \mathscr{C}_c^\infty(\Omega)$. It follows that $P(E) = Q(E)$ for any Borel subset $E \subset \Omega$. ■

When n is even, the operator $\mathscr{L}_n$ is non-local. In particular, the proof of Proposition 8.1 does not apply. We present two approaches attempting to recover a localization result similar to Proposition 8.1. Consider a probability measure P over $\mathbb{R}^n$ with n even. The first idea that naturally comes to mind is to embed P into $\mathbb{R}^{n+1}$ (where $n+1$ is then odd) giving rise to a probability measure P^* supported on the hyperplane $x_{n+1}=0$ of $\mathbb{R}^{n+1}$. Proposition 8.1 therefore applies to P^*. The other approach consists in localizing the operator $(-\Delta)^{1/2}$. For a smooth function u over $\mathbb{R}^n$, computing $(-\Delta)^{1/2}u$ can be done by first solving $-\Delta U=0$ over $\mathbb{R}^{n+1}_+ := \mathbb{R}^n\times(0,\infty)$ subject to the boundary condition $U(x',0)=u(x')$ for any $x'\in\mathbb{R}^n$. One then has

$$((-\Delta)^{1/2}u)(x') = -\lim_{x_{n+1}\to 0}(\partial_{n+1}U)(x',x_{n+1})$$

for any $x'\in\mathbb{R}^n$. This formulation is now local with respect to U since the values of $\partial_{n+1}U$ in some open subset $\Omega\subset\mathbb{R}^{n+1}_+$ depend on the values of U on Ω only. For further details on this method, we refer the reader to [3] and [28].

It turns out that both approaches are equivalent. This is the content of the next proposition, in which we will show that the density f_P of a probability measure P over $\mathbb{R}^n$ (n even) can be recovered through $\lim_{x_{n+1}\to 0}\partial_{n+1}U(x',x_{n+1})$, where $U(x',x_{n+1})$ is issentially equal to

$$(-\Delta)^{\frac{n-2}{2}}(\nabla\cdot R_{P^*})(x',x_{n+1})$$

and solves $-\Delta U=0$ over $\mathbb{R}^{n+1}_+$.

Proposition 8.2. *Let $n\geq 2$ be even and P be a Borel probability measure over $\mathbb{R}^n$. Assume that P admits a density $f_P\in L^1(\mathbb{R}^n)$ with respect to the Lebesgue measure and that $f_P\in\mathscr{C}^{0,\alpha}(\mathbb{R}^n)$ for some $\alpha\in(0,1)$. Let P^* denote the probability measure over $\mathbb{R}^{n+1}$ supported on the hyperplane $x_{n+1}=0$ with density f_P with respect to the n-dimensonal Hausdorff measure $\mathcal{H}_n$. Let Z be a random n-vector with law P and Z^* be a random $(n+1)$-vector with law P^*. Let*

$$U(x) = 2\gamma_{n+1}\mathbb{E}\Big[((-\Delta)^{\frac{n-2}{2}}(\nabla\cdot K_{n+1}))(x-Z^*)\Big]$$

for any $x\in\mathbb{R}^{n+1}_+ := \mathbb{R}^n\times(0,\infty)$ and

$$u(x') = \gamma_n(-\Delta)^{\frac{n-2}{2}}(\nabla\cdot R_P)(x')$$

for any $x'\in\mathbb{R}^n$. We have that $U\in\mathscr{C}^\infty(\mathbb{R}^{n+1}_+)$ and $u\in\mathscr{C}^1(\mathbb{R}^n)$. In addition, the following holds :

1. *for any $x\in\mathbb{R}^{n+1}_+$, $U(x)=2\gamma_{n+1}(-\Delta)^{\frac{n-2}{2}}(\nabla\cdot R_{P^*})(x)$ and $-\Delta U(x)=0$;*
2. *for any $x'\in\mathbb{R}^n$, $U(x',0)=u(x')$ and*

$$f_P(x') = ((-\Delta)^{1/2}u)(x') = \lim_{x_{n+1}\overset{>}{\to}0} -(\partial_{n+1}U)(x',x_{n+1}).$$

In practice, Proposition 8.2 entails that one can recover f_P by applying purely (local) differential operators to the spatial rank associated to P^* instead of P. We summarize this in the following corollary.

Corollary 8.3. *Let $n\geq 2$ be even and P be a Borel probability measure over $\mathbb{R}^n$. Assume that P admits a density $f_P\in L^1(\mathbb{R}^n)$ with respect to the Lebesgue measure and that $f_P\in\mathscr{C}^{0,\alpha}(\mathbb{R}^n)$ for some $\alpha\in(0,1)$. Let P^* denote the probability measure over $\mathbb{R}^{n+1}$ supported on the hyperplane $x_{n+1}=0$ with density f_P with respect to the n-dimensonal Hausdorff measure $\mathcal{H}_n$. Then,*

$$f_P(x') = -2\gamma_{n+1}\lim_{x_{n+1}\overset{>}{\to}0}\partial_{n+1}(-\Delta)^{\frac{n-2}{2}}(\nabla\cdot R_{P^*})(x',x_{n+1})$$

for any $x'\in\mathbb{R}^n$.

Proof of Proposition 8.2. Since P^* admits the null density over the open subset $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0,\infty)$ of $\mathbb{R}^{n+1}$, Proposition 6.8 entails that $R_{P^*} \in \mathscr{C}^n(\mathbb{R}^{n+1}_+)$ and that

$$\partial^\alpha R_{P^*}(x) = \mathbb{E}[(\partial^\alpha K_{n+1})(x - Z^*)]$$

for any $x \in \mathbb{R}^{n+1}_+$ and any $\alpha \in \mathbb{N}^{n+1}$ with $|\alpha| \le n$, where Z^* is a random $(n+1)$-vector with law P^*. Letting

$$U(x) := 2\gamma_{n+1}\mathbb{E}\Big[((-\Delta)^{\frac{n-2}{2}}(\nabla \cdot K_{n+1}))(x - Z^*)\Big]$$

for any $x \in \mathbb{R}^{n+1}_+$, we then have

$$U(x) = 2\gamma_{n+1}(-\Delta)^{\frac{n-2}{2}}(\nabla \cdot R_{P^*})(x)$$

for any $x \in \mathbb{R}^{n+1}_+$. Theorem 6.9 further implies that $-\Delta U(x) = 0$ for any $x \in \mathbb{R}^{n+1}_+$.

Let us show that $U(x', 0) = u(x')$ for any $x' \in \mathbb{R}^n$, where

$$u(x') = \gamma_n(-\Delta)^{\frac{n-2}{2}}(\nabla \cdot R_P)(x').$$

First observe that $R_P \in \mathscr{C}^{n-1}(\mathbb{R}^n)$ and that we have

$$u(x') = \gamma_n\mathbb{E}\Big[((-\Delta)^{\frac{n-2}{2}}(\nabla \cdot K_n))(x' - Z)\Big]$$

for any $x' \in \mathbb{R}^n$, by Proposition 6.8. Let us compute explicitly $(-\Delta)^{\frac{n-2}{2}}(\nabla\cdot K_n)$ and $(-\Delta)^{\frac{n-2}{2}}(\nabla\cdot K_{n+1})$. It is easy to see that

$$(\nabla \cdot K_n)(x') = (n-1)\frac{1}{|x'|}$$

for any $x' \in \mathbb{R}^n \setminus \{0\}$. Easy computations further show that

$$(-\Delta)^\ell \frac{1}{|x'|} = \Lambda_{n,\ell}\frac{1}{|x'|^{2\ell+1}}, \tag{15}$$

for any $x' \in \mathbb{R}^n \setminus \{0\}$, where

$$\Lambda_{n,\ell} = \prod_{j=1}^{\ell}(2j-1)(n-2j-1) \tag{16}$$

for any $1 \le \ell \le \frac{n-2}{2}$. It follows that

$$(-\Delta)^{\frac{n-2}{2}}(\nabla \cdot K_n)(x') = (n-1)\Lambda_{n,\frac{n-2}{2}}\frac{1}{|x'|^{n-1}}$$

for any $x' \in \mathbb{R}^n \setminus \{0\}$. The same computations yield

$$(-\Delta)^{\frac{n-2}{2}}(\nabla \cdot K_{n+1})(x) = n\Lambda_{n+1,\frac{n-2}{2}}\frac{1}{|x|^{n-1}}$$

for any $x \in \mathbb{R}^{n+1} \setminus \{0\}$. Using the fact that

$$\prod_{j=1}^{k}(2j-1) = \frac{(2k)!}{2^k k!}$$

for any integer $k \ge 1$, it is easy to see that

$$\Lambda_{n,\frac{n-2}{2}} = \left(\frac{\Gamma(n-1)}{2^{\frac{n-2}{2}}\Gamma(\frac{n}{2})}\right)^2$$

and $\Lambda_{n+1,\frac{n-2}{2}} = \Gamma(n-1)$. It follows that

$$U(x) = 2\gamma_{n+1} n\Gamma(n-1)\mathbb{E}\Big[\frac{1}{|x-Z^*|^{n-1}}\mathbb{I}[Z \neq x]\Big]$$

for any $x \in \mathbb{R}^{n+1}_+$ and that

$$u(x') = \gamma_n(n-1)\Big(\frac{\Gamma(n-1)}{2^{\frac{n-2}{2}}\Gamma(\frac{n}{2})}\Big)^2\mathbb{E}\Big[\frac{1}{|x'-Z|^{n-1}}\mathbb{I}[Z \neq x']\Big]$$

for any $x' \in \mathbb{R}^n$. In particular, we have that

$$\begin{aligned} U(x',0) &= 2n\gamma_{n+1}\Gamma(n-1)\mathbb{E}\Big[\frac{1}{|x'-Z|^{n-1}}\mathbb{I}[Z \neq x']\Big] \\ &= 2n\gamma_{n+1}\Gamma(n-1) \times \frac{2^{n-2}\Gamma(\frac{n}{2})^2}{\gamma_n(n-1)\Gamma(n-1)^2}u(x') \\ &= \frac{\gamma_{n+1}}{\gamma_n} \times \frac{2^{n-1}n\Gamma(\frac{n}{2})^2}{\Gamma(n)}u(x') \end{aligned}$$

for any $x' \in \mathbb{R}^n$. Using the fact that

$$\Gamma\Big(k+\frac{1}{2}\Big) = \frac{\sqrt{\pi}\ \Gamma(2k+1)}{2^{2k}\Gamma(k+1)} \tag{17}$$

for any $k \in \mathbb{N}$ leads to

$$\frac{\gamma_{n+1}}{\gamma_n} = \frac{\Gamma(n+1)}{2^{n+1}\Gamma(\frac{n}{2}+1)^2} = \frac{n\Gamma(n)}{2^{n+1}(\frac{n}{2}\Gamma(\frac{n}{2}))^2} = \frac{\Gamma(n)}{2^{n-1}n\Gamma(\frac{n}{2})^2}.$$

It follows that $U(x',0) = u(x')$ for any $x' \in \mathbb{R}^n$.

Let us now compute $-\partial_{n+1}U(x', x_{n+1})$ for any $(x', x_{n+1}) \in \mathbb{R}^n \times (0,\infty)$. We have already noticed that

$$\partial_{n+1}U(x) = 2\gamma_{n+1}\mathbb{E}\Big[\partial_{n+1}((-\Delta)^{\frac{n-2}{2}}(\nabla \cdot K_{n+1}))(x-Z^*)\Big]$$

for any $x \in \mathbb{R}^{n+1}_+$. Writing $Z^* = (Z^*_1, \ldots, Z^*_{n+1})$, we then have

$$-(\partial_{n+1}U)(x', x_{n+1}) = 2\gamma_{n+1}\Gamma(n+1)\mathbb{E}\Big[\frac{x_{n+1}-Z^*_{n+1}}{|x-Z^*|^{n+1}}\mathbb{I}[Z \neq x]\Big]$$

for any $(x', x_{n+1}) \in \mathbb{R}^n \times (0,\infty)$. Let us now show that

$$\mathbb{E}\Big[\frac{x_{n+1}-Z^*_{n+1}}{|x-Z^*|^{n+1}}\mathbb{I}[Z \neq x]\Big] \to \frac{\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}f_P(x')$$

as $x_{n+1} \to 0$. Letting $\mathcal{H}_n$ denote the n-dimensional Haussdorf measure in $\mathbb{R}^{n+1}$, we have, for any

$x \in \mathbb{R}^{n+1}_+$

$$\begin{aligned}
&\mathbb{E}\Big[\frac{x_{n+1} - Z^*_{n+1}}{|x - Z^*|^{n+1}} \mathbb{I}[Z \neq x]\Big] \\
&= \int_{\{z_{n+1}=0\}} \frac{x_{n+1} - z_{n+1}}{\Big(|x' - z'|^2 + (x_{n+1} - z_{n+1})^2\Big)^{\frac{n+1}{2}}} f_P(z') d\mathcal{H}_n(z', z_{n+1}) \\
&= \int_{\mathbb{R}^n} \frac{x_{n+1}}{(|x' - z'|^2 + x_{n+1}^2)^{\frac{n+1}{2}}} f_P(z') dz' \\
&= \int_{\mathbb{R}^n} \frac{1}{x_{n+1}^n} \frac{1}{\Big(1 + |\frac{x'-z'}{x_{n+1}}|^2\Big)^{\frac{n+1}{2}}} f_P(z') dz' \\
&= \int_{\mathbb{R}^n} \frac{1}{(1 + |z'|^2)^{\frac{n+1}{2}}} f_P(x' - x_{n+1} z') dz'.
\end{aligned}$$

It follows that

$$\mathbb{E}\Big[\frac{x_{n+1} - Z^*_{n+1}}{|x - Z^*|^{n+1}} \mathbb{I}[Z \neq x]\Big] \to f_P(x') \int_{\mathbb{R}^n} \frac{1}{(1 + |z'|^2)^{\frac{n+1}{2}}}\, dz'$$

as $x_{n+1} \to 0$. Indeed, we have that

$$\begin{aligned}
&\Big| \int_{\mathbb{R}^n} \frac{1}{(1 + |z'|^2)^{\frac{n+1}{2}}} f_P(x' - x_{n+1} z') dz' - f_P(x') \int_{\mathbb{R}^n} \frac{1}{(1 + |z'|^2)^{\frac{n+1}{2}}}\, dz' \Big| \\
&\qquad \leq [f_P]_{\mathscr{C}0,\alpha} |x_{n+1}|^\alpha \int_{\mathbb{R}^n} \frac{|z'|^\alpha}{(1 + |z'|^2)^{\frac{n+1}{2}}}\, dz',
\end{aligned}$$

where $[f_P]_{\mathscr{C}0,\alpha} := \sup_{x \neq y} \frac{|f_P(x) - f_P(y)|}{|x-y|^\alpha}$. The latter converges to 0 as $x_{n+1} \to 0$ because

$$\int_{\mathbb{R}^n} \frac{|z'|^\alpha}{(1 + |z'|^2)^{\frac{n+1}{2}}}\, dz' < \infty$$

since $0 < \alpha < 1$. Furthermore, one can show that

$$\int_{\mathbb{R}^n} \frac{1}{(1 + |z'|^2)^{\frac{n+1}{2}}}\, dz' = S_{n-1} \frac{\sqrt{\pi}\ \Gamma(n/2)}{2\Gamma(\frac{n+1}{2})} = \frac{\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})},$$

where $S_{n-1} = 2\pi^{\frac{n}{2}}/\Gamma(\frac{n}{2})$ is the surface area of the $(n-1)$-dimensional sphere of $\mathbb{R}^n$. We deduce that

$$-(\partial_{n+1} U)(x', x_{n+1}) \to 2\gamma_{n+1} \Gamma(n+1) \frac{\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})} f_p(x')$$

as $x_{n+1} \to 0$. Using again (17), we see that

$$2\gamma_{n+1} \Gamma(n+1) \frac{\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})} = 1.$$

It follows that

$$-(\partial_{n+1} U)(x', x_{n+1}) \to f_P(x')$$

as $x_{n+1} \to 0$. Since $f \in \mathscr{C}^{0,\alpha}(\mathbb{R}^n)$, Theorem 6.10 entails that

$$f_P(x') = \gamma_n (-\Delta)^{1/2} (-\Delta)^{\frac{n-2}{2}} (\nabla \cdot R_P)(x') = ((-\Delta)^{1/2} u)(x')$$

for any $x' \in \mathbb{R}^n$. This concludes the proof. ■

Appendix A: Auxiliary proofs for Section 4.2

Proof of Corollary 4.9. Let $x_0 \in \Omega$ and $r > 0$ be such that $\overline{B(x_0, r)} \subset \Omega$. For any $x \in B_1$, let $\tilde{u}(x) := u(\frac{x-x_0}{r})$ and $\tilde{f}(x) := \frac{1}{r^2} f(\frac{x-x_0}{r})$. Since $\Delta u = f$ in the weak sense in $B(x_0, r)$, a direct computation entails that $\Delta \tilde{u} = \tilde{f}$ in the weak sense in B_1. Since $u \in H^1(B(x_0, r)) \cap L^\infty(B(x_0, r))$ and $f \in \mathscr{C}^{k,\alpha}(B(x_0, r))$, we have that $\tilde{u} \in H^1(B_1) \cap L^\infty(B_1)$ and $\tilde{f} \in \mathscr{C}^{k,\alpha}(B_1)$. It follows from Proposition 4.8 that $\tilde{u} \in \mathscr{C}^{k+2,\alpha}(B_1)$. This implies that $u \in \mathscr{C}^{k+2,\alpha}(B(x_0, r))$. Now let $V \subset \Omega$ be an open subset such that $\overline{V} \subset \Omega$. Then V can be covered by a finite number of balls of the form $B(x_0, r)$ with $\overline{B(x_0, r)} \subset \Omega$. Since u is of class $\mathscr{C}^{k+2,\alpha}$ on each one of these balls, we have that $u \in \mathscr{C}^{k+2,\alpha}(V)$. We conclude that $u \in \mathscr{C}^{k+2,\alpha}_{\text{loc}}(\Omega)$. ■

Appendix B: Auxiliary proofs for Section 5

Proof of Proposition 5.1. Observe that the map $\xi \mapsto (1 + |\xi|)^m |\xi|^{2s} (\mathscr{F} u)(\xi)$ is integrable over $\mathbb{R}^n$ for any $m \geq 0$ since $u \in \mathscr{S}(\mathbb{R}^n)$. This entails that $\mathscr{F}^{-1}(|\xi|^{2s} \mathscr{F} u) \in \mathscr{C}^m(\mathbb{R}^n)$ for any $m \geq 0$, whence $(-\Delta)^s u \in \mathscr{C}^\infty(\mathbb{R}^n)$. It remains to show that (3) holds for any $\alpha \in \mathbb{N}^n$. Observe that

$$\begin{aligned}
\partial^\alpha (-\Delta)^s u &= \partial^\alpha \mathscr{F}^{-1}(|\xi|^{2s} \widehat{u}) \\
&= \mathscr{F}^{-1}\Big((2i\pi\xi)^\alpha |\xi|^{2s} \mathscr{F} u\Big) \\
&= \mathscr{F}^{-1}\Big(|\xi|^{2s} \mathscr{F}(\partial^\alpha u)\Big) \\
&= (-\Delta)^s (\partial^\alpha u)
\end{aligned}$$

for any $\alpha \in \mathbb{N}^n$. Since $\partial^\alpha u$ belongs to $\mathscr{S}(\mathbb{R}^n)$ for any $\alpha \in \mathbb{N}^n$, it is enough to show that

$$\sup_{x \in \mathbb{R}^n} |(1 + |x|^{n+2s})(-\Delta)^s u(x)| \lesssim |u|_{L^1(\mathbb{R}^n)} + \sup_{z \in \mathbb{R}^n} \Big((1 + |z|)^{n+2} |\nabla^2 u(z)|\Big). \tag{18}$$

By simple changes of variable, it is easy to show that

$$(-\Delta)^s u(x) = -\frac{1}{2} c_{n,s} \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}} \, dy$$

for any $x \in \mathbb{R}^n$; see, e.g., Lemma 3.2 in [22]. Notice that this last integral is not singular at $y = 0$ anymore. Indeed, one can easily show that

$$\frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}} = \frac{1}{|y|^{n+2s}} \int_{-1}^{1} (y, \nabla^2 u(x+ty)y) \, dt. \tag{19}$$

The r.h.s. of (19) is then bounded by

$$\frac{|\nabla^2 u|_{L^\infty(\mathbb{R}^n)}}{|y|^{n+2s-2}},$$

which is integrable near the origin. Let us first show that

$$\sup_{x \in \mathbb{R}^n} |(-\Delta)^s u(x)| \lesssim |u|_{L^1(\mathbb{R}^n)} + \sup_{z \in \mathbb{R}^n} \Big((1 + |z|)^{n+2} |\nabla^2 u(z)|\Big). \tag{20}$$

Let us fix $x \in \mathbb{R}^n$ and write

$$-\frac{2}{c_{n,s}}(-\Delta)^s u(x) = \int_{\mathbb{R}^n \setminus B_1} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}\,dy + \int_{B_1} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}\,dy =: I_1(x) + I_2(x).$$

For I_1, we have that

$$|I_1(x)| \leq 4|u|_{L^1(\mathbb{R}^n)}.$$

For I_2, recalling (19), we have that

$$\begin{aligned} |I_2(x)| &\leq \int_{B_1} \frac{1}{|y|^{n+2s-2}} \int_{-1}^{1} |\nabla^2 u(x+ty)|\,dt\,dy \\ &\lesssim \sup_{z\in\mathbb{R}^n} |\nabla^2 u(z)| \\ &\leq \sup_{z\in\mathbb{R}^n} \Big((1+|z|)^{n+2}|\nabla^2 u(z)|\Big). \end{aligned}$$

This yields (20). Let us now show that

$$\sup_{x\in\mathbb{R}^n} \Big(|x|^{n+2s}|(-\Delta)^s u(x)|\Big) \lesssim |u|_{L^1(\mathbb{R}^n)} + \sup_{z\in\mathbb{R}^n} (1+|z|)^{n+2}|\nabla^2 u(z)|. \tag{21}$$

Let us fix $x \in \mathbb{R}^n$ and write

$$-\frac{2}{c_{n,s}}|x|^{n+2s}(-\Delta)^s u(x) = |x|^{n+2s} \int_{\mathbb{R}^n \setminus B_{\frac{1}{2}|x|}} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}\,dy + |x|^{n+2s} \int_{B_{\frac{1}{2}|x|}} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}\,dy =: J_1(x) + J_2(x).$$

For J_1, we have that

$$|J_1(x)| \leq 4|u|_{L^1(\mathbb{R}^n)} 2^{n+2s}.$$

For J_2, recalling (19), we have that

$$\begin{aligned} |J_2(x)| &\leq |x|^{n+2s} \int_{B_{\frac{1}{2}|x|}} \frac{1}{|y|^{n+2s-2}} \int_{-1}^{1} |\nabla^2 u(x+ty)|\,dt\,dy \\ &= \int_{B_{\frac{1}{2}|x|}} \frac{1}{|y|^{n+2s-2}} \int_{-1}^{1} \left(\frac{|x|}{|x+ty|}\right)^{n+2s} |x+ty|^{n+2s} |\nabla^2 u(x+ty)|\,dt\,dy. \end{aligned}$$

For any y such that $|y| \leq \frac{1}{2}|x|$ and any $t \in [-1,1]$, we have that

$$\frac{|x|}{|x+ty|} \leq \frac{|x|}{|x|-|t||y|} \leq 2.$$

For any k, let

$$C_k(u) := \sup_{x\in\mathbb{R}^n} (1+|z|)^k |\nabla^2 u(z)| < \infty.$$

We then have

$$|x+ty|^{n+2s}|\nabla^2 u(x+ty)| \le \frac{C_{N+n+2s}(u)}{(1+|x+ty|)^N} \le \frac{C_{N+n+2s}(u)}{(1+\frac{1}{2}|x|)^N}$$

for any N, any $|y| \le \frac{1}{2}|x|$ and any $t \in [-1,1]$. Let us fix $N = 2-2s$. We then have that

$$|J_2(x)| \le 2^{n+2s} C_{n+2}(u) \frac{1}{(1+\frac{1}{2}|x|)^{2-2s}} \int_{B_{\frac{1}{2}|x|}} \frac{1}{|y|^{n+2s-2}}\, dy.$$

Furthermore, it is easy to see that

$$\int_{B_{\frac{1}{2}|x|}} \frac{1}{|y|^{n+2s-2}}\, dy \lesssim |x|^{2-2s}.$$

It follows that

$$\sup_{x\in\mathbb{R}^n} |J_2(x)| \lesssim \sup_{x\in\mathbb{R}^n} (1+|z|)^{n+2}|\nabla^2 u(z)|.$$

We deduce that

$$\sup_{x\in\mathbb{R}^n} \Big(|x|^{n+2s}|(-\Delta)^s u(x)|\Big) \lesssim |u|_{L^1(\mathbb{R}^n)} + \sup_{x\in\mathbb{R}^n} (1+|z|)^{n+2}|\nabla^2 u(z)|,$$

which establishes (21). Putting (20) and (21) together yields (18), which concludes the proof. ■

Proof of Proposition 5.4. 1. Since $u \in L^2(\mathbb{R}^n)$, we have that $u \in \mathscr{S}_s(\mathbb{R}^n)'$ so that $(-\Delta)^s u$ is a well-defined tempered distribution. Let $\psi \in \mathscr{S}(\mathbb{R}^n)$. We have that

$$\langle (-\Delta)^s u, \psi\rangle = \langle u, (-\Delta)^s \psi\rangle = (2\pi)^{2s} \int_{\mathbb{R}^n} u(x) \mathscr{F}^{-1}(|\xi|^{2s}\mathscr{F}\psi)(x)\, dx.$$

Since $u \in L^2(\mathbb{R}^n)$ and $|\xi|^{2s}\mathscr{F}\psi \in L^2(\mathbb{R}^n)$, we have

$$\langle (-\Delta)^s u, \psi\rangle = (2\pi)^{2s} \int_{\mathbb{R}^n} (\mathscr{F}^{-1}u)(\xi)|\xi|^{2s}(\mathscr{F}\psi)(\xi)\, d\xi$$

by interchanging the inverse Fourier transform under the integral. Since $u \in H^{2s}(\mathbb{R}^n)$, we have that $|\xi|^{2s}\mathscr{F}^{-1}u \in L^2(\mathbb{R}^n)$. Since $\psi \in L^2(\mathbb{R}^n)$, interchanging the Fourier transform again yields

$$\langle (-\Delta)^s u, \psi\rangle = (2\pi)^{2s} \int_{\mathbb{R}^n} \mathscr{F}(|\xi|^{2s}(\mathscr{F}^{-1}u))(x)\psi(x)\, dx.$$

Observing that

$$\mathscr{F}(|\xi|^{2s}(\mathscr{F}^{-1}u)) = \mathscr{F}^{-1}(|\xi|^{2s}(\mathscr{F}u))$$

yields

$$\langle (-\Delta)^s u, \psi\rangle = \int_{\mathbb{R}^n} (2\pi)^{2s}\mathscr{F}^{-1}(|\xi|^{2s}(\mathscr{F}u))(x)\psi(x)\, dx.$$

Since $\psi \in \mathscr{S}(\mathbb{R}^n)$ was arbitrary, the conclusion follows.

2. Observe that $|\xi|^{2s}\mathscr{F}u \in \mathscr{S}(\mathbb{R}^n)'$ since $|\xi|^{2s}\mathscr{F}u \in L^1(\mathbb{R}^n)$. In particular, $\mathscr{F}^{-1}(|\xi|^{2s}\mathscr{F}u) \in \mathscr{S}(\mathbb{R}^n)'$ as well. Let $\psi \in \mathscr{S}(\mathbb{R}^n)$. We have

$$\begin{aligned}\left\langle (2\pi)^{2s}\mathscr{F}^{-1}(|\xi|^{2s}\mathscr{F}u), \psi\right\rangle &= \left\langle |\xi|^{2s}\mathscr{F}u, (2\pi)^{2s}\mathscr{F}^{-1}\psi\right\rangle \\ &= \int_{\mathbb{R}^n} |\xi|^{2s}\mathscr{F}u(\xi)(2\pi)^{2s}(\mathscr{F}^{-1}\psi)(\xi)\, d\xi \\ &= \int_{\mathbb{R}^n} \mathscr{F}u(\xi)\mathscr{F}^{-1}\Big((2\pi)^{2s}\mathscr{F}\left(|\xi|^{2s}\mathscr{F}^{-1}\psi\right)\Big)(\xi)\, d\xi \\ &= \int_{\mathbb{R}^n} \mathscr{F}u(\xi)\mathscr{F}^{-1}((-\Delta)^s\psi)(\xi)\, d\xi.\end{aligned}$$

Since $\psi \in \mathscr{S}(\mathbb{R}^n)$, we have that $(-\Delta)^s\psi \in \mathscr{S}_s(\mathbb{R}^n)$. In particular, $(-\Delta)^s\psi \in L^1(\mathbb{R}^n)$. Since $\mathscr{F}u \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $|\xi|^{2s}\mathscr{F}u$, it is clear that $\mathscr{F}u \in L^1(\mathbb{R}^n)$. By "hat skipping", we therefore have that

$$\left\langle (2\pi)^{2s}\mathscr{F}^{-1}(|\xi|^{2s}\mathscr{F}u), \psi\right\rangle = \int_{\mathbb{R}^n} \mathscr{F}^{-1}(\mathscr{F}u)(x)((-\Delta)^s\psi)(x)\, dx.$$

Observe that since the tempered distribution $\mathscr{F}u$ is a function of $L^1(\mathbb{R}^n)$, we have that $\mathscr{F}^{-1}(\mathscr{F}u)$ is a continuous function. It is further easy to see that $\mathscr{F}^{-1}(\mathscr{F}u) = u$ almost everywhere on $\mathbb{R}^n$, although u might not be integrable. We conclude that

$$\left\langle (2\pi)^{2s}\mathscr{F}^{-1}(|\xi|^{2s}\mathscr{F}u), \psi\right\rangle = \int_{\mathbb{R}^n} u(x)((-\Delta)^s\psi)(x)\, dx = \langle (-\Delta)^s u, \psi\rangle$$

for any $\psi \in \mathscr{S}(\mathbb{R}^n)$.

3. This is proved in [28]; see Proposition 2.4. ■

Appendix C: Auxiliary proofs for Section 6

Proof of Lemma 6.2. The proof follows the same lines as [16] and [1] (see Equation (1.1.1) in [16] and Theorem 56 in [1]). Let $g \in L^1(\mathbb{R}^n)$ be such that $g(x) = h(|x|)$ for any $x \in \mathbb{R}^n$. We first prove that

$$\widehat{g}(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}|\xi|^n}\int_0^\infty r^{\frac{n}{2}} h\Big(\frac{r}{2\pi|\xi|}\Big) J_{\frac{n-2}{2}}(r)\, dr \tag{22}$$

for any $\xi \in \mathbb{R}^n \setminus \{0\}$, where J_ν is the Bessel function of the first kind of order ν. Let us therefore fix $\xi \in \mathbb{R}^n \setminus \{0\}$. Observe that

$$\widehat{g}(\xi) = \int_{\mathbb{R}^n} g(x)e^{-2i\pi(x,\xi)}\, dx = \int_{\mathbb{R}^n} g(x)e^{-2i\pi(x,O\xi)}\, dx$$

for any $n \times n$ orthogonal matrix, since $g(x) = h(|x|)$ for any x. Let us then assume that $\xi = |\xi|(1, 0, \ldots, 0)$ and let us compute

$$\begin{aligned}\int_{\mathbb{R}^n} g(x)e^{-2i\pi(x,\xi)}\, dx &= \int_{\mathbb{R}^n} h(|x|)e^{-2i\pi|\xi|x_1}\, dx \\ &= \int_{\mathbb{R}} e^{-2i\pi|\xi|x_1}\Big(\int_{\mathbb{R}^{n-1}} h\Big(\sqrt{x_1^2+|y|^2}\Big)\, dy\Big)\, dx_1 \\ &= S_{n-2}\int_{\mathbb{R}} e^{-2i\pi|\xi|x_1}\Big(\int_0^\infty t^{n-2}h\Big(\sqrt{x_1^2+t^2}\Big)\, dt\Big)\, dx_1,\end{aligned}$$

where $S_{n-2} = 2\pi^{\frac{n-1}{2}}/\Gamma(\frac{n-1}{2})$ is the surface area of the $(n-2)$-dimensional unit sphere in $\mathbb{R}^{n-1}$. Let us now write (x_1, t) into spherical coordinates in the plane

$$(x_1, t) = (r\cos\theta, r\sin\theta),$$

with $r \in [0,\infty)$ and $\theta \in [0,\pi]$, since $t > 0$. We then have that

$$\begin{aligned}\widehat{g}(\xi) &= S_{n-2}\int_0^\infty \left(\int_0^\pi e^{-2i\pi|\xi|r\cos\theta}(r\sin\theta)^{n-2}h(r)\, r d\theta\right) dr \\ &= S_{n-2}\int_0^\infty r^{n-1}h(r)\left(\int_0^\pi e^{-2i\pi|\xi|r\cos\theta}(\sin\theta)^{n-2}d\theta\right) dr.\end{aligned}$$

Let us now express $\int_0^\pi e^{-2i\pi|\xi|r\cos\theta}(\sin\theta)^{n-2}d\theta$ in terms of Bessel functions. For any $\nu \in \mathbb{C}$ with $\mathrm{Re}(\nu) > -1/2$, the Bessel function of the first kind of order ν can be computed as

$$J_\nu(x) = \frac{(\frac{x}{2})^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_{-1}^1 (1-t^2)^{\nu-\frac{1}{2}}\cos(xt)\, dt = \frac{(\frac{x}{2})^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_{-1}^1 (1-t^2)^{\nu-\frac{1}{2}}e^{-ixt}\, dt,$$

for any $x \in \mathbb{R}$ (see (10.9.4) in [24]). Substituting $t = \cos\theta$ leads to

$$J_\nu(x) = \frac{(\frac{x}{2})^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_0^\pi (\sin\theta)^{2\nu}e^{-ix\cos\theta}\, d\theta$$

for any $x \in \mathbb{R}$. It follows that

$$\int_0^\pi e^{-2i\pi|\xi|r\cos\theta}(\sin\theta)^{n-2}d\theta = \frac{\sqrt{\pi}\,\Gamma(\frac{n-1}{2})}{(\pi|\xi|r)^{\frac{n-2}{2}}}J_{\frac{n-2}{2}}(2\pi|\xi|r)$$

for any $r > 0$. We deduce that

$$\begin{aligned}\widehat{g}(\xi) &= \frac{2\pi}{|\xi|^{\frac{n-2}{2}}}\int_0^\infty r^{\frac{n}{2}}h(r)J_{\frac{n-2}{2}}(2\pi|\xi|r)\, dr \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}|\xi|^n}\int_0^\infty r^{\frac{n}{2}}h\Big(\frac{r}{2\pi|\xi|}\Big)J_{\frac{n-2}{2}}(r)\, dr,\end{aligned}$$

for any $\xi \in \mathbb{R}^n \setminus \{0\}$, which yields (22). Let $\alpha \in (0, \frac{n+1}{2})$. For any $k \in \mathbb{N}$, let $g_{k,\alpha}(x) := |x|^{\alpha-n}\mathbb{I}[0 < |x| < k]$ for any $x \in \mathbb{R}^n$, and $h_{k,\alpha}(t) = t^{\alpha-n}\mathbb{I}[0 < t < k]$ for any $t > 0$ so that $g_{k,\alpha}(x) = h_{k,\alpha}(|x|)$ for any $x \in \mathbb{R}^n$. For any k, we have that $g_{k,\alpha} \in L^1(\mathbb{R}^n)$. For any $\xi \in \mathbb{R}^n$, applying (22) to $g_{k,\alpha}$ leads to

$$\widehat{g_{k,\alpha}}(\xi) = \frac{1}{(2\pi)^{\alpha-\frac{n}{2}}|\xi|^\alpha}\int_0^{2\pi|\xi|k} r^{\alpha-\frac{n}{2}}J_{\frac{n-2}{2}}(r)\, dr.$$

According to (10.22.43) in [24], this integral converges to

$$\int_0^\infty r^{\alpha-\frac{n}{2}}J_{\frac{n-2}{2}}(r)\, dr = 2^{\alpha-\frac{n}{2}}\frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})}$$

as $k \to \infty$, since $\alpha \in (0, \frac{n+1}{2})$. It follows that

$$\lim_{k\to\infty}\widehat{g_{k,\alpha}}(\xi) = \frac{\pi^{\frac{n}{2}-\alpha}\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})}\,\frac{1}{|\xi|^\alpha}$$

for any $\xi \in \mathbb{R}^n \setminus \{0\}$, with $|\widehat{g_{k,\alpha}}(\xi)| \lesssim \frac{1}{|\xi|^\alpha}$ for any $\xi \in \mathbb{R}^n \setminus \{0\}$, uniformly in k. Let $\psi \in \mathscr{S}(\mathbb{R}^n)$. Observe that

$$|\widehat{g_{k,\alpha}}(\xi)\psi(\xi)| \lesssim \frac{|\psi(\xi)|}{|\xi|^\alpha}$$

uniformly in k, where $|\xi|^{-\alpha}|\psi(\xi)| \in L^1(\mathbb{R}^n)$ since $\alpha < n$ (recall that $\alpha < \frac{n+1}{2}$ and that $n \geq 2$) and $\psi \in \mathscr{S}(\mathbb{R}^n)$. Since $\widehat{g_{k,\alpha}}$ converges almost everywhere over $\mathbb{R}^n$, the dominated convergence theorem entails that

$$\int_{\mathbb{R}^n} \widehat{g_{k,\alpha}}(\xi)\psi(\xi)\, d\xi \to \frac{\pi^{\frac{n}{2}-\alpha}\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})} \int_{\mathbb{R}^n} \frac{1}{|\xi|^\alpha}\psi(\xi)\, d\xi$$

as $k \to \infty$. On the other hand, for any k, we have that

$$\int_{\mathbb{R}^n} \widehat{g_{k,\alpha}}(\xi)\psi(\xi)\, d\xi = \int_{\mathbb{R}^n} g_{k,\alpha}(\xi)\widehat{\psi}(\xi)\, d\xi,$$

since $g_{k,\alpha} \in L^1(\mathbb{R}^n)$. Similarly, we have that $|g_{k,\alpha}(x)\widehat{\psi}(x)| \leq |x|^{\alpha-n}|\psi(x)| \in L^1(\mathbb{R}^n)$, uniformly in k, and we have that

$$\int_{\mathbb{R}^n} g_{k,\alpha}(\xi)\widehat{\psi}(\xi)\, d\xi \to \int_{\mathbb{R}^n} \frac{1}{|x|^{n-\alpha}}\widehat{\psi}(\xi)\, d\xi$$

by dominated convergence. It follows that

$$\int_{\mathbb{R}^n} \frac{1}{|x|^{n-\alpha}}\widehat{\psi}(\xi)\, d\xi = \frac{\pi^{\frac{n}{2}-\alpha}\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})} \int_{\mathbb{R}^n} \frac{1}{|\xi|^\alpha}\psi(\xi)\, d\xi$$

for any $\psi \in \mathscr{S}\mathbb{R}^n)$. We deduce that

$$\mathscr{F}\Big(\frac{1}{|x|^{n-\alpha}}\Big)(\xi) = \frac{\pi^{\frac{n}{2}-\alpha}\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})}\frac{1}{|\xi|^\alpha} \tag{23}$$

in $\mathscr{S}(\mathbb{R}^n)'$ for any $\alpha \in (0, \frac{n+1}{2})$. Taking the inverse Fourier transform on both sides of (23) yields

$$\mathscr{F}\Big(\frac{1}{|x|^{\alpha}}\Big)(\xi) = \frac{\Gamma(\frac{n-\alpha}{2})}{\pi^{\frac{n}{2}-\alpha}\Gamma(\frac{\alpha}{2})}\frac{1}{|\xi|^{n-\alpha}} \tag{24}$$

in $\mathscr{S}(\mathbb{R}^n)'$ for any $\alpha \in (0, \frac{n+1}{2})$. Now let $\beta \in (\frac{n-1}{2}, n)$ and let us write $\beta = n-\alpha$ for some $\alpha \in (0, \frac{n+1}{2})$. Then, (23) yields

$$\mathscr{F}\Big(\frac{1}{|x|^{\beta}}\Big)(\xi) = \frac{\pi^{\frac{n}{2}-\alpha}\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})}\frac{1}{|\xi|^\alpha} = \frac{\Gamma(\frac{n-\beta}{2})}{\pi^{\frac{n}{2}-\beta}\Gamma(\frac{\beta}{2})}\frac{1}{|\xi|^{n-\beta}} \tag{25}$$

in $\mathscr{S}(\mathbb{R}^n)'$ for any $\beta \in (\frac{n-1}{2}, n)$. Putting (24) and (25) together yields the conclusion for any $\alpha \in (0, \frac{n+1}{2}) \cup (\frac{n-1}{2}, n) = (0, n)$. ■

Proof of Lemma 6.3. Let $\psi \in \mathscr{S}(\mathbb{R}^n)$ and let us compute

$$\begin{aligned}\big\langle \nabla(1/|x|^{n-1}), \psi\big\rangle &= -\big\langle 1/|x|^{n-1}, \nabla\psi\big\rangle \\ &= -\int_{\mathbb{R}^n} \frac{1}{|x|^{n-1}}\nabla\psi(x)\, dx \\ &= -\lim_{\substack{R\to\infty \\ \eta\to 0}} \int_{B_R\setminus B_\eta} \frac{1}{|x|^{n-1}}\nabla\psi(x)\, dx,\end{aligned}$$

where the last equality follows by dominated convergence since $x \mapsto 1/|x|^{n-1}$ is integrable near the origin in $\mathbb{R}^n$ and $\nabla\psi \in L^1(\mathbb{R}^n)$ since $\psi \in \mathscr{S}(\mathbb{R}^n)$. Fix $0 < \eta < R < \infty$ We have By Green's formula

$$\int_{B_R \setminus B_\eta} \frac{1}{|x|^{n-1}} \nabla\psi(x)\, dx = -\int_{B_R \setminus B_\eta} \nabla\Big(\frac{1}{|x|^{n-1}}\Big)\psi(x)\, dx$$
$$+ \int_{\partial B_R} \frac{1}{|x|^{n-1}} \psi(x) \frac{x}{|x|}\, d\sigma_R(x) - \int_{\partial B_\eta} \frac{1}{|x|^{n-1}} \psi(x) \frac{x}{|x|}\, d\sigma_\eta(x),$$

where σ_r is the surface area measure on the sphere of radius r. Letting $u = x/R$, we find

$$\int_{\partial B_R} \frac{1}{|x|^{n-1}} \psi(x) \frac{x}{|x|}\, d\sigma_R(x) = R^{n-1} \int_{S^{n-1}} \frac{1}{R^{n-1}} \psi(Ru) u\, d\sigma_1(u).$$

Since ψ is bounded and $\psi(x) \to 0$ as $|x| \to \infty$, the latter converges to 0 as $R \to \infty$. Similarly, we have

$$\int_{\partial B_\eta} \frac{1}{|x|^{n-1}} \psi(x) \frac{x}{|x|}\, d\sigma_\eta(x) = \eta^{n-1} \int_{S^{n-1}} \frac{1}{\eta^{n-1}} \psi(\eta u) u\, d\sigma_1(u).$$

As $\eta \to 0$, the last integral converges to $\psi(0) \int_{S^{n-1}} u\, d\sigma_1(u) = 0$. It follows that

$$\begin{aligned}
\big\langle \nabla(1/|x|^{n-1}), \psi \big\rangle &= \lim_{\substack{R\to\infty \\ \eta\to 0}} \int_{B_R \setminus B_\eta} \nabla\Big(\frac{1}{|x|^{n-1}}\Big)\psi(x)\, dx \\
&= -(n-1) \lim_{\substack{R\to\infty \\ \eta\to 0}} \int_{B_R \setminus B_\eta} \frac{x}{|x|^{n+1}} \psi(x)\, dx \\
&= -(n-1) \lim_{\eta\to 0} \int_{\mathbb{R}^n \setminus B_\eta} \frac{x}{|x|^{n+1}} \psi(x)\, dx \\
&= -(n-1)\ \text{P.V.}\Big(\frac{x}{|x|^{n+1}}\Big).
\end{aligned}$$

■

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