Supplement to "Multivariate ρ-Quantiles: a Spatial Approach"*

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In this supplement to Konen and Paindaveine (2021), we prove all the results stated in the paper.

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Below, $(n.m)$ (resp., (S.$n.m$)) denotes Equation m of Section n from the main manuscript (resp., from this supplemental article). Section S.n, Theorem $m.n$ or Lemma S.$m.n$ are used in the same way.

S.1. Some preliminary results

In this section, we provide results that will be repeatedly used in the proofs of the next sections.

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Lemma S.1.1. *Let ν be a finite measure on $\mathbb{R}^d$. Let (μ_n) be a sequence in $\mathbb{R}^d$ that either (i) is such that $\|\mu_n\|$ diverges to infinity or (ii) converges to $\mu \in \mathbb{R}^d$ but satisfies $\mu_n \neq \mu$ for any n. Then $\nu(\{\mu_n\}) \to 0$ as $n \to \infty$.*

Proof of Lemma S.1.1. Assume, ad absurdum, that $\nu(\{\mu_n\})$ does not converge to 0 as $n \to \infty$. Then there exist $\varepsilon > 0$ and a subsequence (n_ℓ) such that $\nu(\{\mu_{n_\ell}\}) \geq \varepsilon$ for any ℓ. By using Assumptions (i)–(ii), we may assume, up to extraction of a further subsequence, that (μ_{n_ℓ}) has pairwise different terms. We then have

$$\nu(\mathbb{R}^d) \geq \sum_{\ell=0}^{\infty} \nu(\{\mu_{n_\ell}\}) \geq \sum_{\ell=0}^{\infty} \varepsilon = \infty,$$

which is a contradiction. □

Lemma S.1.2. *Let $v, w \in \mathbb{R}^d \setminus \{0\}$. Then*

$$\left\| \frac{v}{\|v\|} - \frac{w}{\|w\|} \right\| \leq 2 \min\left(\frac{\|v-w\|}{\|v\|}, \frac{\|v-w\|}{\|w\|} \right).$$

Proof of Lemma S.1.2. A direct computation provides

$$\left\| \frac{v}{\|v\|} - \frac{w}{\|w\|} \right\| \leq \left\| \frac{\|w\|v - \|v\|v - \|v\|(w-v)}{\|v\|\|w\|} \right\| \leq \frac{\big|\|w\| - \|v\|\big|}{\|w\|} + \frac{\|w-v\|}{\|w\|} \leq 2\frac{\|v-w\|}{\|w\|}\cdot$$

Since one may interchange the roles of v and w in these inequalities, the result follows. □

The following result is a version of the mean-value theorem for one-sided derivatives; see, e.g., Leise and Cohen (2007).

Lemma S.1.3. *Let $f : [a,b] \to \mathbb{R}$ be a continuous function. (i) Assume that f is left-differentiable on (a,b), with left-derivative f'_-. Then,*

$$f'_-(c_1) \leq \frac{f(b) - f(a)}{b-a} \leq f'_-(c_2)$$

for some $c_1, c_2 \in (a,b)$. (ii) Assume that f is right-differentiable on (a,b), with right-derivative f'_+. Then,

$$f'_+(c_1) \leq \frac{f(b) - f(a)}{b-a} \leq f'_+(c_2)$$

for some $c_1, c_2 \in (a,b)$.

We end this section with a result that states structural properties of any loss function $\rho \in \mathcal{C}$.

Lemma S.1.4. *Let $\rho \in \mathcal{C}$. Then, (i) $t\psi_-(t) \geq \rho(t)$ for any $t > 0$. (ii) $t \mapsto \psi_-(t)$, $t \mapsto \psi_+(t)$, and $t \mapsto \rho(t)/t$ are monotone non-decreasing on $(0, \infty)$. (iii) $\psi_-(t) > 0$ for any $t > 0$. (iv) ρ is monotone strictly increasing on $[0, \infty)$.*

Proof of Lemma S.1.4. (i) Convexity of ρ implies that, for any $t > 0$,

$$\frac{\rho(t)}{t} = \frac{\rho(t) - \rho(0)}{t - 0} \leq \psi_-(t),$$

which establishes the result. (ii) This trivially follows from the convexity of ρ. (iii) Assume ad absurdum that $\psi_-(t_0) = 0$ for some $t_0 > 0$. Then, Parts (i)–(ii) of the result imply that $\psi_-(t) = 0$ for any $t \in (0, t_0)$. Lemma S.1.3(i) then entails that $\rho(t_0) = 0$, which contradicts the fact that $\rho(t) = 0$ only for $t = 0$. (iv) The result follows from Part (iii) and Lemma S.1.3(i). □

S.2. Proofs for Section 2

We first prove Part (i) of Theorem 2.1.

Proof of Theorem 2.1(i). We need to show that

$$\mathcal{I} := \int_{\mathbb{R}^d} \left| H^\rho_{\alpha,u}(z - \mu) - H^\rho_{\alpha,u}(z) \right| dP(z) < \infty \tag{S.2.1}$$

for any $\mu \in \mathbb{R}^d$. Since the result trivially holds for $\mu = 0$, we may assume that $\mu \neq 0$. Recalling that $\rho(0) = 0$, note then that

$$\begin{aligned}
\mathcal{I} \quad &\leq \quad \int_{\mathbb{R}^d} \left| \rho(\|z - \mu\|) - \rho(\|z\|) \right| \left(1 + \alpha \frac{u'(z - \mu)}{\|z - \mu\|} \right) \xi_{z,\mu} \, dP(z) \\
&\qquad + \int_{\mathbb{R}^d} \rho(\|z\|) \left| \left(1 + \alpha \frac{u'(z - \mu)}{\|z - \mu\|} \right) \xi_{z,\mu} - \left(1 + \alpha \frac{u'z}{\|z\|} \right) \xi_{z,0} \right| dP(z) \\
&\leq \quad (1 + \alpha) \int_{\mathbb{R}^d} \left| \rho(\|z - \mu\|) - \rho(\|z\|) \right| dP(z) \\
&\qquad + (1 + \alpha) \rho(\|\mu\|) P[\{\mu\}] + \alpha \int_{\mathbb{R}^d \setminus \{0, \mu\}} \rho(\|z\|) \left| \frac{u'(z - \mu)}{\|z - \mu\|} - \frac{u'z}{\|z\|} \right| dP(z) \\
&=: \quad (1 + \alpha) \mathcal{I}_1 + (1 + \alpha) \rho(\|\mu\|) P[\{\mu\}] + \alpha \mathcal{I}_2,
\end{aligned}$$

say. Lemma S.1.3 implies that there exists c between $\|z-\mu\|$ and $\|z\|$ such that $|\rho(\|z-\mu\|)-\rho(\|z\|)| \leq \psi_-(c)|\|z-\mu\|-\|z\||$. Since ψ_- is monotone non-decreasing and nonnegative over $(0,\infty)$ (Lemma S.1.4), we obtain

$$\begin{aligned}\big|\rho(\|z-\mu\|)-\rho(\|z\|)\big| &\leq \|\mu\|\{\psi_-(\|z-\mu\|)+\psi_-(\|z\|)\}\\ &\leq \|\mu\|\{\psi_-(\|z-\mu\|+\delta_\mu)+\psi_-(\|z\|+\delta_0)\},\end{aligned}$$

where δ_μ and δ_0 are as in is as in (4). The finiteness of $\mathcal{I}_1$ therefore follows from the assumption that ρ belongs to $\mathcal{C}$. Now, Lemma S.1.2 shows that any $z \in \mathbb{R}^d \setminus \{0,\mu\}$ satisfies

$$\left|\frac{u'(z-\mu)}{\|z-\mu\|}-\frac{u'z}{\|z\|}\right| \leq \left\|\frac{z-\mu}{\|z-\mu\|}-\frac{z}{\|z\|}\right\| \leq 2\frac{\|\mu\|}{\|z\|}.$$

Since convexity of ρ entails that $\rho(\|z\|)/\|z\| \leq \psi_-(\|z\|) \leq \psi_-(\|z\|+\delta_0)$ for any $z \in \mathbb{R}^d \setminus \{0\}$, we then have

$$\mathcal{I}_2 \leq 2\|\mu\| \int_{\mathbb{R}^d\setminus\{0,\mu\}} \frac{\rho(\|z\|)}{\|z\|}\,dP(z) \leq 2\|\mu\| \int_{\mathbb{R}^d} \psi_-(\|z\|+\delta_0)\,dP(z) < \infty.$$

This proves (S.2.1), hence establishes the result. □

The proof of Theorem 2.1(ii) requires Lemmas S.2.1–S.2.2 below.

Lemma S.2.1. *Let $\rho \in \mathcal{C}$ and $P \in \mathcal{P}_d^\rho$. Then, for any $(\mu_0,\alpha_0,u_0) \in \mathbb{R}^d \times [0,1] \times \mathcal{S}^{d-1}$, the map $(\mu,\alpha,u) \mapsto M^\rho_{\alpha,u}(\mu)$ is Lipschitz at (μ_0,α_0,u_0), in the sense that there exist a positive constant C and a neighbourhood $\mathcal{N}$ of (μ_0,α_0,u_0) such that, for any $(\mu,\alpha,u) \in \mathcal{N}$,*

$$\big|M^\rho_{\alpha,u}(\mu)-M^\rho_{\alpha_0,u_0}(\mu_0)\big| \leq C\big\{\|\mu-\mu_0\|+|\alpha-\alpha_0|+\|u-u_0\|\big\}.$$

In particular, the map $(\mu,\alpha,u) \mapsto M^\rho_{\alpha,u}(\mu)$ is continuous over $\mathbb{R}^d \times [0,1] \times \mathcal{S}^{d-1}$.

Proof of Lemma S.2.1. Fix $(\mu_0,\alpha_0,u_0) \in \mathbb{R}^d \times [0,1] \times \mathcal{S}^{d-1}$. We need to prove that there exist a neighborhood $\mathcal{N}$ of (μ_0,α_0,u_0) and a positive constant C such that

$$\big|M^\rho_{\alpha,u}(\mu)-M^\rho_{\alpha_0,u_0}(\mu_0)\big| \leq C\big\{\|\mu-\mu_0\|+|\alpha-\alpha_0|+\|u-u_0\|\big\}$$

for any $(\mu,\alpha,u) \in \mathcal{N} \cap (\mathbb{R}^d \times [0,1] \times \mathcal{S}^{d-1})$. To that end, write, with $v=\alpha u$ and $v_0=\alpha_0 u_0$,

$$\begin{aligned}&M^\rho_{\alpha,u}(\mu)-M^\rho_{\alpha_0,u_0}(\mu_0)\\ &\qquad= \int_{\mathbb{R}^d} \big\{[H^\rho_{\alpha,u}(z-\mu)-H^\rho_{\alpha_0,u_0}(z-\mu_0)]-[H^\rho_{\alpha,u}(z)-H^\rho_{\alpha_0,u_0}(z)]\big\}\,dP(z)\\ &\qquad= \int_{\mathbb{R}^d} \{T_1(z)+T_2(z)+T_3(z)\}\,dP(z),\end{aligned}$$

where

$$T_1(z) := \{\rho(\|z-\mu\|) - \rho(\|z-\mu_0\|)\}\bigg(1+\frac{v'(z-\mu)}{\|z-\mu\|}\bigg)\xi_{z,\mu},$$

$$T_2(z) := \{\rho(\|z-\mu_0\|) - \rho(\|z\|)\}\bigg\{\bigg(1+\frac{v'z}{\|z\|}\bigg)\xi_{z,0} - \bigg(1+\frac{v_0'z}{\|z\|}\bigg)\xi_{z,0}\bigg\},$$

and

$$T_3(z) := \rho(\|z-\mu_0\|)\bigg\{\bigg(1+\frac{v'(z-\mu)}{\|z-\mu\|}\bigg)\xi_{z,\mu} - \bigg(1+\frac{v_0'(z-\mu_0)}{\|z-\mu_0\|}\bigg)\xi_{z,\mu_0} \\ -\bigg(1+\frac{v'z}{\|z\|}\bigg)\xi_{z,0} + \bigg(1+\frac{v_0'z}{\|z\|}\bigg)\xi_{z,0}\bigg\}.$$

Assume that there exists a positive constant C such that

$$|T_\ell(z)| \le C\{1+\psi_-(\|z\|)+\psi_-(\|z-\mu\|)+\psi_-(\|z-\mu_0\|)\}\{\|\mu-\mu_0\|+\|v-v_0\|\} \quad \text{(S.2.2)}$$

for any $\ell = 1,2,3$ and any $z \in \mathbb{R}^d$ (in the rest of this proof, C may change from line to line). Monotonicity of ψ_- then ensures that, for μ close enough to $\mu_0 \in \mathbb{R}^d$, we have (with δ_μ as in (4))

$$|T_\ell(z)| \le C\{1+\psi_-(\|z\|+\delta_0)+2\psi_-(\|z-\mu_0\|+\delta_{\mu_0})\}\{\|\mu-\mu_0\|+\|v-v_0\|\}$$

for any $\ell = 1,2,3$ and any $z \in \mathbb{R}^d$. Since $\rho \in \mathcal{C}$ and $\|v-v_0\| \le |\alpha-\alpha_0| + \|u-u_0\|$, this provides

$$\begin{aligned} |M^\rho_{\alpha,u}(\mu) - M^\rho_{\alpha_0,u_0}(\mu_0)| &\le \sum_{\ell=1}^{3} \int_{\mathbb{R}^d} |T_\ell(z)|\}\, dP(z) \\ &\le C(\|\mu-\mu_0\| + |\alpha-\alpha_0| + \|u-u_0\|), \end{aligned}$$

as was to be shown. It is therefore sufficient to prove that there indeed exists a positive constant C such that (S.2.2) holds for any $\ell = 1,2,3$ and any $z \in \mathbb{R}^d$.

Using Lemma S.1.3 and the fact that ψ_- is non-decreasing, we obtain that, for some c between $\|z-\mu\|$ and $\|z-\mu_0\|$,

$$\begin{aligned} |T_1(z)| &\le 2\psi_-(c)\big|\|z-\mu\| - \|z-\mu_0\|\big| \\ &\le 2\{\psi_-(\|z-\mu\|)+\psi_-(\|z-\mu_0\|)\}\|\mu-\mu_0\|, \end{aligned}$$

which shows that (S.2.2) holds for $T_1(z)$. Noting that $T_2(z)$ rewrites

$$T_2(z) = \{\rho(\|z-\mu_0\|) - \rho(\|z\|)\}\frac{(v-v_0)'z}{\|z\|}\xi_{z,0},$$

we obtain in the same way (here, c is between $\|z-\mu_0\|$ and $\|z\|$)

$$\begin{aligned}|T_2(z)| &\leq \psi_-(c)\big|\|z-\mu_0\|-\|z\|\big|\|v-v_0\| \\ &\leq \{\psi_-(\|z-\mu_0\|+\psi_-(\|z\|)\}\|\mu_0\|\|v-v_0\|,\end{aligned}$$

so that (S.2.2) also holds for $T_2(z)$. We may thus focus on $T_3(z)$. Note that, if $z \notin \{0,\mu,\mu_0\}$, then Lemma S.1.2 yields

$$\begin{aligned}|T_3(z)| &\leq \rho(\|z-\mu_0\|)\left|\frac{v'(z-\mu)}{\|z-\mu\|}-\frac{v_0'(z-\mu_0)}{\|z-\mu_0\|}-\frac{v'z}{\|z\|}+\frac{v_0'z}{\|z\|}\right| \\ &\leq \rho(\|z-\mu_0\|)\left\|(v-v_0)'\left(\frac{z-\mu_0}{\|z-\mu_0\|}-\frac{z}{\|z\|}\right)+v'\left(\frac{z-\mu}{\|z-\mu\|}-\frac{z-\mu_0}{\|z-\mu_0\|}\right)\right\| \\ &\leq \rho(\|z-\mu_0\|)\left(2\|v-v_0\|\frac{\|\mu_0\|}{\|z-\mu_0\|}+2\|v\|\frac{\|\mu-\mu_0\|}{\|z-\mu_0\|}\right) \\ &\leq C\psi_-(\|z-\mu_0\|)\big(\|v-v_0\|+\|\mu-\mu_0\|\big),\end{aligned}$$

where we used the fact that $\rho(t)/t \leq \psi_-(t)$ for any $t \in (0,\infty)$. Obviously, if $z=\mu_0$, then $T_3(z)=0$, whereas if $z=\mu(\neq \mu_0)$, then

$$|T_3(z)| \leq 4\rho(\|\mu-\mu_0\|) \leq 4\psi_-(\|\mu-\mu_0\|)\|\mu-\mu_0\| = 4\psi_-(\|z-\mu_0\|)\|\mu-\mu_0\|.$$

Finally, if $z=0 \notin \{\mu,\mu_0\}$, then Lemma S.1.2 provides

$$\begin{aligned}T_3(z) &\leq \rho(\|\mu_0\|)\left|\frac{v_0'\mu_0}{\|\mu_0\|}-\frac{v'\mu}{\|\mu\|}\right| \leq \rho(\|\mu_0\|)\left|(v_0-v)'\frac{\mu_0}{\|\mu_0\|}+v'\left(\frac{\mu_0}{\|\mu_0\|}-\frac{\mu}{\|\mu\|}\right)\right| \\ &\leq \rho(\|\mu_0\|)\left(\|v-v_0\|+\left\|\frac{\mu}{\|\mu\|}-\frac{\mu_0}{\|\mu_0\|}\right\|\right) \leq \rho(\|\mu_0\|)\left(\|v-v_0\|+2\frac{\|\mu-\mu_0\|}{\|\mu_0\|}\right) \\ &\leq \rho(\|\mu_0\|)\|v-v_0\|+2\psi_-(\|\mu_0\|)\|\mu-\mu_0\| \\ &\leq C\bigg(\|v-v_0\|+\|\mu-\mu_0\|\bigg),\end{aligned}$$

which shows that (S.2.2) holds for $T_3(z)$, too. The result is proved. □

Note that if the assumption that $P \in \mathcal{P}_d^\rho$ is reinforced into the assumption that

$$\int_{\mathbb{R}^d} \psi_-(\|z\|+c)\,dP(z) < \infty$$

for any $c>0$, then the proof of Lemma S.2.1 shows that $(\mu,\alpha,u) \mapsto M^\rho_{\alpha,u}(\mu)$ is actually Lipschitz over $K \times [0,1] \times \mathcal{S}^{d-1}$ for any compact set $K \subset \mathbb{R}^d$. In particular, for $\rho(t)=t^p$

with $p \geq 1$, if P has finite moments of order $p-1$, then $(\mu, \alpha, u) \mapsto M^{\rho}_{\alpha,u}(\mu)$ is not only locally Lipschtiz (Lemma S.2.1) but satisfies this global Lipschtiz property over compact sets.

The next result states a coercivity property for the objective function.

Lemma S.2.2. *Let $\rho \in \mathcal{C}$ and $P \in \mathcal{P}^{\rho}_{d}$. Fix sequences $(\alpha_\ell) \in [0,1]$, (u_ℓ) in $\mathcal{S}^{d-1}$, and (μ_ℓ) in $\mathbb{R}^d$ such that $c := \limsup_{\ell\to\infty} \alpha_\ell u'_\ell(\mu_\ell/\|\mu_\ell\|)\xi_{\mu_\ell,0} < 1$ and $\|\mu_\ell\| \to \infty$. Then,*

$$\liminf_{\ell\to\infty} \frac{M^{\rho}_{\alpha_\ell,u_\ell}(\mu_\ell)}{\|\mu_\ell\|} > 0.$$

In particular, $M^{\rho}_{\alpha_\ell,u_\ell}(\mu_\ell) \to \infty$.

Proof of Lemma S.2.2. Of course, we may assume without any loss of generality that $\mu_\ell \neq 0$ for any ℓ. Note that if $P[\{0\}] = 1$, then we have $M^{\rho}_{\alpha_\ell,u_\ell}(\mu_\ell) = H^{\rho}_{\alpha_\ell,u_\ell}(-\mu_\ell) = \rho(\|\mu_\ell\|)(1 - \alpha_\ell u'_\ell \mu_\ell/\|\mu_\ell\|)$, so that the fact that $t \mapsto \rho(t)/t$ is monotone non-decreasing (Lemma S.1.4) yields

$$\liminf_{\ell\to\infty} \frac{M^{\rho}_{\alpha_\ell,u_\ell}(\mu_\ell)}{\|\mu_\ell\|} \geq (1-c) \liminf_{\ell\to\infty} \frac{\rho(\|\mu_\ell\|)}{\|\mu_\ell\|} \geq (1-c)\rho(1) > 0.$$

We may thus assume that $P[\{0\}] < 1$. Write then

$$\frac{M^{\rho}_{\alpha_\ell,u_\ell}(\mu_\ell)}{\|\mu_\ell\|} = \mathcal{I}_1(\alpha_\ell, u_\ell, \mu_\ell) + \mathcal{I}_2(\alpha_\ell, u_\ell, \mu_\ell) + \mathcal{I}_3(\alpha_\ell, u_\ell, \mu_\ell), \tag{S.2.3}$$

with

$$\mathcal{I}_1(\alpha, u, \mu) = \int_{\mathbb{R}^d} \frac{\rho(\|z-\mu\|) - \rho(\|z\|)}{\|\mu\|} \left(1 + \alpha \frac{u'(z-\mu)}{\|z-\mu\|}\right) \xi_{z,\mu}\, dP(z),$$

$$\begin{aligned}\mathcal{I}_2(\alpha, u, \mu) &= \int_{\mathbb{R}^d\setminus\{0,\mu\}} \frac{\rho(\|z\|)}{\|\mu\|} \left\{ \left(1 + \alpha \frac{u'(z-\mu)}{\|z-\mu\|}\right) \xi_{z,\mu} - \left(1 + \alpha \frac{u'z}{\|z\|}\right) \xi_{z,0} \right\} dP(z) \\ &= \int_{\mathbb{R}^d\setminus\{0\}} \alpha \frac{\rho(\|z\|)}{\|\mu\|} \left(\frac{u'(z-\mu)}{\|z-\mu\|} - \frac{u'z}{\|z\|} \right) \xi_{z,\mu}\, dP(z),\end{aligned} \tag{S.2.4}$$

and

$$\begin{aligned}\mathcal{I}_3(\alpha, u, \mu) &= \int_{\{0,\mu\}} \frac{\rho(\|z\|)}{\|\mu\|} \left\{ \left(1 + \alpha \frac{u'(z-\mu)}{\|z-\mu\|}\right) \xi_{z,\mu} - \left(1 + \alpha \frac{u'z}{\|z\|}\right) \xi_{z,0} \right\} dP(z) \\ &= -\frac{\rho(\|\mu\|)}{\|\mu\|} \left(1 + \alpha \frac{u'\mu}{\|\mu\|}\right) P[\{\mu\}] \xi_{\mu,0}.\end{aligned}$$

Consider first $\mathcal{I}_1(\alpha_\ell, u_\ell, \mu_\ell)$. If $\|z-\mu_\ell\| > \|z\|$, then Lemma S.1.3 provides

$$\rho(\|z-\mu_\ell\|) - \rho(\|z\|) \geq \psi_-(c_\ell^-)\big(\|z-\mu_\ell\| - \|z\|\big) \geq \psi_-(\|z\|)\big(\|z-\mu_\ell\| - \|z\|\big),$$

with c_ℓ^- between $\|z-\mu_\ell\|$ and $\|z\|$, whereas if $\|z-\mu_\ell\| < \|z\|$, then the same result yields

$$\rho(\|z\|) - \rho(\|z-\mu_\ell\|) \leq \psi_-(c_\ell^+)\big(\|z\| - \|z-\mu_\ell\|\big) \leq \psi_-(\|z\|)\big(\|z\| - \|z-\mu_\ell\|\big),$$

still with c_ℓ^+ between $\|z-\mu_\ell\|$ and $\|z\|$. Thus, for any $z \in \mathbb{R}^d$, we have

$$\rho(\|z-\mu_\ell\|) - \rho(\|z\|) \geq \psi_-(\|z\|)\big(\|z-\mu_\ell\| - \|z\|\big),$$

which provides

$$\mathcal{I}_1(\alpha_\ell, u_\ell, \mu_\ell) \geq \tilde{\mathcal{I}}_1(\alpha_\ell, u_\ell, \mu_\ell) = \int_{\mathbb{R}^d} h_{\alpha_\ell, u_\ell, \mu_\ell}(z)\, dP(z),$$

where we let

$$h_{\alpha,u,\mu}(z) = \psi_-(\|z\|)\frac{\|z-\mu\| - \|z\|}{\|\mu\|}\bigg(1 + \alpha\frac{u'(z-\mu)}{\|z-\mu\|}\bigg)\xi_{z,\mu}.$$

Since the function $z \mapsto |h_{\alpha_\ell,u_\ell,\mu_\ell}(z)|$ is upper-bounded by the P-integrable function $z \mapsto 2\psi_-(\|z\|)$ uniformly in ℓ, Fatou's lemma shows that

$$\begin{aligned}
\liminf_{\ell\to\infty} \mathcal{I}_1(\alpha_\ell, u_\ell, \mu_\ell) &\geq \liminf_{\ell\to\infty} \tilde{\mathcal{I}}_1(\alpha_\ell, u_\ell, \mu_\ell) \geq \int_{\mathbb{R}^d} \liminf_{\ell\to\infty} h_{\alpha_\ell,u_\ell,\mu_\ell}(z)\, dP(z) \\
&\geq \int_{\mathbb{R}^d} \liminf_{\ell\to\infty} \bigg(\psi_-(\|z\|)\frac{\|z-\mu_\ell\| - \|z\|}{\|\mu_\ell\|}\xi_{z,\mu_\ell}\bigg) \liminf_{\ell\to\infty} \bigg(1 + \alpha_\ell\frac{u_\ell'(z-\mu_\ell)}{\|z-\mu_\ell\|}\bigg) dP(z) \\
&= (1-c)\int_{\mathbb{R}^d} \psi_-(\|z\|)\, dP(z) \geq (1-c)\int_{\mathbb{R}^d\setminus\{0\}} \psi_-(\|z\|)\, dP(z) > 0,
\end{aligned}$$

where the last inequality follows from the fact that $P[\mathbb{R}^d \setminus \{0\}] > 0$ and that $\psi_-(t) > 0$ for any $t > 0$ (Lemma S.1.4).

Let us turn to $\mathcal{I}_2(\alpha_\ell, u_\ell, \mu_\ell)$. For any $z \in \mathbb{R}^d \setminus \{0\}$, we have $\rho(\|z\|) \leq \|z\|\psi_-(\|z\|)$ and

$$\bigg|\frac{u_\ell'(z-\mu_\ell)}{\|z-\mu_\ell\|} - \frac{u_\ell' z}{\|z\|}\bigg|\xi_{z,\mu_\ell} \leq \frac{2\|\mu_\ell\|}{\|z\|}$$

(Lemma S.1.2). Therefore, the absolute value of the integrand in (S.2.4) is upper-bounded by the P-integrable function $z \mapsto 2\psi_-(\|z\|)$. Since this function does not depend on ℓ, Lebesgue's Dominated Convergence Theorem (DCT) shows that

$$\lim_{\ell\to\infty} \mathcal{I}_2(\alpha_\ell, u_\ell, \mu_\ell) = \int_{\mathbb{R}^d\setminus\{0\}} \lim_{\ell\to\infty} \alpha_\ell \frac{\rho(\|z\|)}{\|\mu_\ell\|}\bigg(\frac{u_\ell'(z-\mu_\ell)}{\|z-\mu_\ell\|} - \frac{u_\ell' z}{\|z\|}\bigg)\xi_{z,\mu_\ell}\, dP(z) = 0.$$

Finally, using again the identity $\rho(\|z\|) \leq \|z\|\psi_-(\|z\|)$, we obtain

$$\liminf_{\ell\to\infty} \mathcal{I}_3(\alpha_\ell, u_\ell, \mu_\ell) \geq -2 \limsup_{\ell\to\infty} \Big(\psi_-(\|\mu_\ell\|) P[\{\mu_\ell\}]\Big) = 0,$$

where the limsup vanishes by Lemma S.1.1 applied to the measure attributing to any d-Borel set B the measure $\nu(B) = \int_B \psi_-(\|z\|) dP(z)$ (which is finite by assumption). Therefore,

$$\liminf_{\ell\to\infty} \frac{M^\rho_{\alpha_\ell,u_\ell}(\mu_\ell)}{\|\mu_\ell\|} \geq \liminf_{\ell\to\infty} \mathcal{I}_1(\alpha_\ell, u_\ell, \mu_\ell) > 0,$$

which establishes the result. □

We can now prove Part (ii) of Theorem 2.1.

Proof of Theorem 2.1(ii). Theorem 2.1(i) ensures that the map $\mu \mapsto M^\rho_{\alpha,u}(\mu)$ is well-defined for any $\mu \in \mathbb{R}^d$. Pick $R > 0$ such that $M^\rho_{\alpha,u}(\mu) > M^\rho_{\alpha,u}(0)$ for any $\|\mu\| > R$ (existence follows from Lemma S.2.2 since $\alpha < 1$). From continuity (Lemma S.2.1), the map $\mu \mapsto M^\rho_{\alpha,u}(\mu)$ admits a minimum over the compact set $K = \{\mu \in \mathbb{R}^d : \|\mu\| \leq R\}$. Since $M^\rho_{\alpha,u}(\mu) > M^\rho_{\alpha,u}(0)$ for any $\|\mu\| > R$, this minimum over K is actually a minimum over $\mathbb{R}^d$. □

S.3. Proofs for Section 3

Proof of Lemma 3.1. (i) By definition, $\mathcal{C}_0 \subseteq \mathcal{C}$. Now, fix $\rho \in \mathcal{C}$. Since ρ is monotone non-decreasing and convex, we have $\rho(\|(1-\lambda)x+\lambda y\|) \leq \rho((1-\lambda)\|x\|+\lambda\|y\|) \leq (1-\lambda)\rho(\|x\|)+\lambda\rho(\|y\|)$ for any $x, y \in \mathbb{R}^d$ and any $\lambda \in [0,1]$. This shows that $z \mapsto H^\rho_{0,u}(z) = \rho(\|z\|)$ is convex for any $u \in \mathcal{S}^{d-1}$, hence that $\rho \in \mathcal{C}_0$. Therefore, we also have $\mathcal{C} \subseteq \mathcal{C}_0$. (ii) For any $\alpha \in [0,1]$, define

$$V_\alpha := \big\{\rho \in \mathcal{C} : g^\rho_\alpha(x,y) := a(x,y) - \alpha\|V(x,y)\| \geq 0 \ \forall x, y \in \mathbb{R}^d\big\},$$

with

$$a(x,y) := \rho(\|x\|) + \rho(\|y\|) - 2\rho\bigg(\frac{\|x+y\|}{2}\bigg)$$

and

$$V(x,y) := \rho(\|x\|)\frac{x}{\|x\|}\xi_{x,0} + \rho(\|y\|)\frac{y}{\|y\|}\xi_{y,0} - 2\rho\bigg(\frac{\|x+y\|}{2}\bigg)\frac{x+y}{\|x+y\|}\xi_{x+y,0}.$$

Note that $H^\rho_{\alpha,u}(x)+H^\rho_{\alpha,u}(y)-2H^\rho_{\alpha,u}((x+y)/2) = a(x,y)+\alpha u'V(x,y)$. Since we trivially have that $V_{\alpha_2} \subseteq V_{\alpha_1}$ for any $\alpha_1 < \alpha_2$, it is sufficient to prove that $\mathcal{C}_\alpha = V_\alpha$ for any $\alpha \in [0,1]$.

Fix first $\rho \in \mathcal{C}_\alpha$. Then, for any $x, y \in \mathbb{R}^d$ and any $u \in \mathcal{S}^{d-1}$, we have $a(x,y) + \alpha u'V(x,y) \geq 0$. If $V(x,y) = 0$, then $g_\alpha^\rho(x,y) = a(x,y) \geq 0$ since $x \mapsto \rho(\|x\|)$ is convex (see Part (i) of the result), whereas if $V(x,y) \neq 0$, then taking $u_0 = -V(x,y)/\|V(x,y)\|$ yields $g_\alpha^\rho(x,y) = a(x,y) + \alpha u_0' V(x,y) \geq 0$ since $x \mapsto H^\rho_{\alpha,u_0}(x)$ is convex. Thus, $g_\alpha^\rho(x,y) \geq 0$ for any $x, y \in \mathbb{R}^d$, so that $\rho \in V_\alpha$. Now, fix $\rho \in V_\alpha$. Then, the Cauchy–Schwarz inequality ensures that, for any $x, y \in \mathbb{R}^d$ and any $u \in \mathcal{S}^{d-1}$, one has $a(x,y) + \alpha u'V(x,y) \geq g_\alpha^\rho(x,y) \geq 0$. This shows that, for any $u \in \mathcal{S}^{d-1}$, the map $H^\rho_{\alpha,u}$ is midpoint convex, hence convex (from continuity). In other words, $\rho \in \mathcal{C}_\alpha$. □

The following result plays a key role in the proof of Theorem 3.1.

Lemma S.3.1. *Let $\rho \in \mathcal{C}$ and fix $t \in \mathcal{D}_\rho$. Then, the Hessian matrix $\nabla^2 H^\rho_{1,u}(x)$ is positive semi-definite for any $u \in \mathcal{S}^{d-1}$ and any $x \in \mathbb{R}^d$ with $\|x\| = t$ if and only if the second-order derivative of $s \mapsto s^2/\rho(s)$ at t is nonpositive.*

Proof of Lemma S.3.1. Throughout the proof, we write $y = x/\|x\|$. For any $u, v \in \mathcal{S}^{d-1}$ and any x such that $\|x\| = t$, Lemma S.5.1 (whose proof is independent) entails that

$$
\begin{aligned}
& v'\nabla^2 H^\rho_{1,u}(x)v \\
&\quad = \frac{t^2\psi_-'(t) - 2t\psi_-(t) + 2\rho(t)}{t^2}(1+u'y)(v'y)^2 + \frac{\rho(t)}{t^2}(1-(v'y)^2) \\
&\qquad + \frac{t\psi_-(t) - \rho(t)}{t^2}\big\{(1+u'y)(1-(v'y)^2) + 2(v'y)^2 + 2(v'y)(u'v)\big\} \\
&\quad = \psi_-'(t)(v'y)^2 + \frac{\psi_-(t)}{t}(1-(v'y)^2) \\
&\qquad + u'\Bigg\{\psi_-'(t)(v'y)^2 y + \frac{t\psi_-(t) - \rho(t)}{t^2}(1-(v'y)^2)y \\
&\qquad\qquad + \frac{2(t\psi_-(t) - \rho(t))}{t^2}(v'y)\{v - (v'y)y\}\Bigg\} = a + u'\{(a-b)y + cy^\perp\},
\end{aligned}
$$

with

$$
a = \psi_-'(t)(v'y)^2 + \frac{\psi_-(t)}{t}(1-(v'y)^2), \quad b = \frac{\rho(t)}{t^2}(1-(v'y)^2),
$$
$$
c = \frac{2(t\psi_-(t) - \rho(t))}{t^2}(v'y), \quad \text{and} \quad y^\perp = (I_d - yy')v.
$$

Clearly, $v'\nabla^2 H^\rho_{1,u}(ty)v \geq 0$ for any $u, v, y \in \mathcal{S}^{d-1}$ if and only if $a - \|(a-b)y + cy^\perp\| \geq 0$ for any $v, y \in \mathcal{S}^{d-1}$. Since $y'y^\perp = 0$ and since $a \geq 0$ (recall that ρ is convex and non-decreasing), this is in turn equivalent to requiring that $2ab - b^2 - c^2\|y^\perp\|^2 \geq 0$, that is,

$$2\Big(\psi'_-(t)(v'y)^2 + \frac{\psi_-(t)}{t}(1-(v'y)^2)\Big)\frac{\rho(t)}{t^2}$$
$$-\frac{(\rho(t))^2}{t^4}(1-(v'y)^2) - \frac{4(t\psi_-(t)-\rho(t))^2}{t^4}(v'y)^2 \geq 0$$

for any $v, y \in \mathcal{S}^{d-1}$ with $v \notin \{\pm y\}$, or again that

$$(1-(v'y)^2)\bigg(2\psi_-(t) - \frac{\rho(t)}{t}\bigg)\frac{\rho(t)}{t^3} + (v'y)^2\frac{2}{t^4}\Big(t^2\rho(t)\psi'_-(t) - 2(t\psi_-(t)-\rho(t))^2\Big) \geq 0$$

for any such v, y. Since the assumptions on ρ imply that $2\psi_-(t) - \rho(t)/t \geq \psi_-(t) \geq 0$, this is equivalent to requiring that $t^2\rho(t)\psi'_-(t) - 2(t\psi_-(t)-\rho(t))^2 \geq 0$, that is,

$$-(\rho(t))^3\frac{d^2}{ds^2}\frac{s^2}{\rho(s)}\bigg|_{s=t} = t^2\rho(t)\psi'_-(t) + 4t\rho(t)\psi_-(t) - 2t^2(\psi_-(t))^2 - 2(\rho(t))^2 \geq 0.$$

This establishes the result. □

The proof of Theorem 3.1 further requires the following result.

Lemma S.3.2. *Let $g : [a,b] \to \mathbb{R}$ be a continuous function that is differentiable in $[a,b] \setminus \{s_1, \ldots, s_k\}$, with $s_0 := a < s_1 < \ldots < s_k < b =: s_{k+1}$ ($k \in \mathbb{N}$). Assume that, for any $j = 1, \ldots, k+1$, g' is monotone non-decreasing in (s_{j-1}, s_j), and that, for any $j = 1, \ldots, k$,*

$$\ell_j := \lim_{s \overset{<}{\to} s_j} g'(s) \quad \text{and} \quad r_j := \lim_{s \overset{>}{\to} s_j} g'(s)$$

exist, are finite, and satisfy $\ell_j \leq r_j$. Then, g is convex on $[a,b]$.

Proof of Lemma S.3.2. For any $j = 1, \ldots, k$, L'Hospital's rule ensures that

$$\lim_{s \overset{<}{\to} s_j} \frac{g(s) - g(s_j)}{s - s_j} = \ell_j,$$

which shows that g is actually left-differentiable in (a,b). The monotonicity assumption on g' and the assumption that $\ell_j \leq r_j$ for any $j = 1, \ldots, k$, then entail that, g'_-, the left-derivative of g, is monotone non-decreasing in $(0,1)$.

Now, assume ad absurdum that g is not convex $[a,b]$. Then, there exist $u, v, w \in [a,b]$, with $u < v < w$, such that

$$\frac{g(v) - g(u)}{v - u} > \frac{g(w) - g(v)}{w - v}.$$

Lemma S.1.3(i) then ensures that there exist $\eta \in (u,v)$ and $\xi \in (v,w)$ such that $g'(\eta) > g'(\xi)$. Therefore, g'_- is not monotone non-decreasing in $(0,1)$, a contradiction. □

Proof of Theorem 3.1. Assume first that $\alpha_\rho = 1$, so that $H^\rho_{1,u}$ is convex for any $u \in \mathcal{S}^{d-1}$. Fix a positive integer k and let $\mathcal{U}_k = (t_{k-1}, t_k)$, where the t_ℓ's are the endpoints of the intervals on which ρ is twice continuously differentiable (see the beginning of Section 2). Thus, $v'\nabla^2 H^\rho_{1,u}(x)v \geq 0$ for any $u, v \in \mathcal{S}^{d-1}$ and $x \in E_k = \{x \in \mathbb{R}^d : \|x\| \in \mathcal{U}_k\}$. Letting $f(t) := \frac{d}{dt}(t^2/\rho(t))$ for any $t \in \mathcal{U}_k$, Lemma S.3.1 then yields that f is monotone non-increasing on $\mathcal{U}_k$. Now, convexity of ρ implies that

$$\lim_{t \overset{<}{\to} t_k} f(t) = \frac{2t_k\rho(t_k) - t_k^2\psi_-(t_k)}{(\rho(t_k))^2} \geq \frac{2t_k\rho(t_k) - t_k^2\psi_+(t_k)}{(\rho(t_k))^2} = \lim_{t \overset{>}{\to} t_k} f(t)$$

(recall that ψ_- and ψ_+ are the left- and right-derivatives of ρ, respectively). Since this holds for any positive integer k, we conclude that f is monotone non-increasing on $(0,\infty)$, hence that $t \mapsto t^2/\rho(t)$ is concave on $(0,\infty)$.

Assume now that $t \mapsto t^2/\rho(t)$ is concave on $(0,\infty)$. Fix $x, y \in \mathbb{R}^d$ linearly independent and let $\Gamma(s) = (1-s)x + sy$, $s \in [0,1]$. Then, $\Gamma(s) \neq 0$ for any $s \in [0,1]$. Let $s_1, \ldots, s_r$ be the values in $(0,1)$ for which $\|\Gamma(s)\| \notin \mathcal{D}_\rho$. Since $s \mapsto \|\Gamma(s)\|^2$ is convex, r is finite. Letting $s_0 = 0$ and $s_{r+1} = 1$,

$$s \mapsto g(s) = H^\rho_{1,u}(\Gamma(s)) = \rho(\|\Gamma(s)\|)\left(1 + \frac{u'\Gamma(s)}{\|\Gamma(s)\|}\right)$$

is twice continuously differentiable on (s_j, s_{j+1}) for any $j = 0, 1, \ldots, r$. Fix then such a value of j. Since $\Gamma''(s) = 0$ for any $s \in (s_j, s_{j+1})$, we have

$$g''(s) = (\Gamma'(s))'\nabla^2 H^\rho_{1,u}(\Gamma(s))\Gamma'(s) \geq 0$$

for any $s \in (s_j, s_{j+1})$ (non-negativity follows from Lemma S.3.1 since the concavity assumption ensures that $\frac{d^2}{dt^2}(t^2/\rho(t)) \leq 0$ for any $t \in \mathcal{D}_\rho$). This implies that g is convex on any interval (s_j, s_{j+1}). Now, noting that the derivative of $s \mapsto \|\Gamma(s)\|$ is $(y-x)'\Gamma(s)/\|\Gamma(s)\|$, we have, for any $j = 1, \ldots, r$,

$$\begin{aligned}
&\lim_{s \overset{>}{\to} s_j} g'(s) - \lim_{s \overset{<}{\to} s_j} g'(s) \\
&\qquad = (\psi_+(\|\Gamma(s_j)\|) - \psi_-(\|\Gamma(s_j)\|))\frac{(y-x)'\Gamma(s_j)}{\|\Gamma(s_j)\|}\left(1 + \frac{u'\Gamma(s_j)}{\|\Gamma(s_j)\|}\right)
\end{aligned}$$

if $(y-x)'\Gamma(s_j) \geq 0$, and

$$\begin{aligned}&\lim_{s \overset{>}{\to} s_j} g'(s) - \lim_{s \overset{<}{\to} s_j} g'(s) \\ &\qquad = (\psi_-(\|\Gamma(s_j)\|) - \psi_+(\|\Gamma(s_j)\|)) \frac{(y-x)'\Gamma(s_j)}{\|\Gamma(s_j)\|}\left(1 + \frac{u'\Gamma(s_j)}{\|\Gamma(s_j)\|}\right)\end{aligned}$$

if $(y-x)'\Gamma(s_j) < 0$. In both cases, convexity of ρ implies that

$$\lim_{s \overset{>}{\to} s_j} g'(s) \geq \lim_{s \overset{<}{\to} s_j} g'(s)$$

for any $j = 1, \ldots, r$. Therefore, Lemma S.3.2 entails that g is convex on $[0,1]$. This yields that

$$H^{\rho}_{1,u}((1-\lambda)x + \lambda y) \leq (1-\lambda)H^{\rho}_{1,u}(x) + \lambda H^{\rho}_{1,u}(y), \quad \forall \lambda \in (0,1), \tag{S.3.5}$$

for any $x, y \in \mathbb{R}^d$ that are linearly independent. Since (S.3.5) also holds if x, y are linearly dependent (it is easy to check that, for any $y \in \mathcal{S}^{d-1}$, the map $t \mapsto H^{\rho}_{1,u}(ty) = \rho(|t|)(1 + \mathrm{Sign}(t)u'y)$ is convex over $\mathbb{R}$), we conclude that $H^{\rho}_{1,u}$ is convex on $\mathbb{R}^d$ for any $u \in \mathcal{S}^{d-1}$. The result then follows from Lemma 3.1. □

Proof of Theorem 3.2. Assume first that ρ is twice continuously differentiable on $(0,\infty)$, that is, assume that $\mathcal{D}_\rho = (0,\infty)$. Fix $t \in \mathcal{D}_\rho$ such that $(t^2/\rho(t))'' > 0$. Recall that convexity of ρ implies that $t\psi_-(t) \geq \rho(t)$. If $t\psi_-(t) = \rho(t)$, then we have

$$\left(\frac{t^2}{\rho(t)}\right)'' = -\frac{t^2\rho(t)\psi_-'(t) + 4t\rho(t)\psi_-(t) - 2t^2(\psi_-(t))^2 - 2(\rho(t))^2}{(\rho(t))^3} = -\frac{t^2\psi_-'(t)}{(\rho(t))^2},$$

which implies that $\psi_-'(t) < 0$. Since this is incompatible with the convexity of ρ, we must have $t\psi_-(t) > \rho(t)$. In other words,

$$\begin{aligned}\mathcal{D}^{\mathrm{cv}}_\rho &= \{t \in \mathcal{D}_\rho : (t^2/\rho(t))'' > 0\} \\ &= \{t \in \mathcal{D}_\rho : t\psi_-(t) > \rho(t) \text{ and } (t^2/\rho(t))'' > 0\},\end{aligned}$$

which allows us to partition $\mathcal{D}_\rho$ into $\mathcal{D}^{\mathrm{cv}}_\rho \cup \mathcal{E}_\rho \cup \mathcal{F}_\rho$, with

$$\mathcal{E}_\rho := \{t \in \mathcal{D}_\rho : t\psi_-(t) > \rho(t) \text{ and } (t^2/\rho(t))'' \leq 0\}$$

and

$$\mathcal{F}_\rho := \{t \in \mathcal{D}_\rho : t\psi_-(t) = \rho(t)\}.$$

Now, note that, for $t \in \mathcal{F}_\rho$, Lemma S.5.1 provides

$$v'\nabla^2 H^\rho_{\alpha,u}(ty)v = \psi'_-(t)(1+\alpha u'y)(v'y)^2 + \frac{\rho(t)}{t^2}(1-(v'y)^2) \geq 0, \tag{S.3.6}$$

for any $\alpha \in [0,1]$ and $u, v, y \in \mathcal{S}^{d-1}$. For $t \in \mathcal{D}_\rho \setminus \mathcal{F}_\rho$, the quantity q_t is well-defined, and we have $(p_t, q_t) \in (1,\infty) \times [0,\infty)$; here, p_t and q_t refer to the quantities defined in the statement of the theorem. For $t \in \mathcal{D}_\rho \setminus \mathcal{F}_\rho$, Lemma S.5.1 then yields

$$\begin{aligned}\frac{t^2}{\rho(t)} v'\nabla^2 H^\rho_{\alpha,u}(ty)v &= (p_t-1)(q_t-2)(1+\alpha u'y)(v'y)^2 + 1 - (v'y)^2 \\ &+ (p_t-1)\big\{(1+\alpha u'y)(1-(v'y)^2) + 2(v'y)^2 + 2\alpha(v'y)(u'v)\big\} =: g_{t,\alpha}(u,v,y).\end{aligned}$$

With an obvious abuse of notation, write

$$\begin{aligned}g_{t,\alpha}(\theta,\omega) &= (p_t-1)(q_t-2)(1+\alpha\cos\theta)(\cos\omega)^2 + 1 - (\cos\omega)^2 \\ &+ (p_t-1)\big\{(1+\alpha\cos\theta)(1-(\cos\omega)^2) + 2(\cos\omega)^2 + 2\alpha(\cos\omega)\cos(\theta-\omega)\big\},\end{aligned}$$

where $\omega = \arccos(v'y) \in [0,\pi]$ is the angle between v and y and $\theta = s_{u,v,y}\arccos(u'y) \in [-\pi,\pi]$ is the signed angle between u and y; here, $s_{u,v,y} = 1$ (resp., $s_{u,v,y} = -1$) if, in the plane containing u, v, y, we have that u and v are not separated (resp., are separated) by the line through $\pm y$.

Let

$$\tilde\alpha_\rho := \inf_{t \in \mathcal{D}^{\rm cv}_\rho} \tilde\alpha_{t,\rho}, \quad \text{with } \tilde\alpha_{t,\rho} := \sqrt{\frac{q_t(4p_t^2 - 4p_t - q_t)}{4(p_t-1)^2(q_t+1)}}$$

(the assumption that $t \mapsto t^2/\rho(t)$ is not concave on $(0,\infty)$ ensures that $\mathcal{D}^{\rm cv}_\rho$ is non-empty, so that $\tilde\alpha_\rho$ is well-defined). We will show that

- (i)$_a$ for any $t \in \mathcal{D}^{\rm cv}_\rho$ and (θ,ω), $g_{t,\tilde\alpha_\rho}(\theta,\omega) \geq 0$;
- (i)$_b$ for any $t \in \mathcal{E}_\rho$ and (θ,ω), $g_{t,\tilde\alpha_\rho}(\theta,\omega) \geq 0$;
- (ii) for any $\alpha > \tilde\alpha_\rho$, there exist $t \in \mathcal{D}^{\rm cv}_\rho$ and (θ,ω) such that $g_{t,\alpha}(\theta,\omega) < 0$.

Jointly with (S.3.6), (i)$_a$–(i)$_b$ establish that $\alpha_\rho \geq \tilde\alpha_\rho$, whereas (ii) entails that $\alpha_\rho \leq \tilde\alpha_\rho$. Therefore, it is sufficient to prove (i)$_a$–(ii). To do so, fix $t \in \mathcal{D}_\rho \setminus \mathcal{F}_\rho$ and write

$$g_{t,\alpha}(\theta,\omega) = i_t(\omega) + \alpha s_t(\theta,\omega),$$

where the intercept function $i_t(\omega)$ and slope function $s_t(\theta,\omega)$ are defined as

$$i_t(\omega) := p_t + (p_t q_t - p_t - q_t)(\cos\omega)^2$$

and

$$s_t(\theta,\omega) := (p_t-1)\{(1+(q_t-1)(\cos\omega)^2)\cos\theta + \sin(2\omega)\sin\theta\}.$$

Note that $i_t(\omega) \geq \min(p_t, p_t + (p_t q_t - p_t - q_t)) = \min(p_t, (p_t-1)q_t) \geq 0$. Actually, if $i_t(\omega) = 0$, then $q_t = 0$ and $(\cos\omega)^2 = 1$, which yields $g_{t,\alpha}(\theta,\omega) = 0$ for any $\alpha \in [0,1]$ and $(\theta,\omega) \in [-\pi,\pi]\times[0,\pi]$. We may thus ignore this case when investigating when $g_{t,\alpha}(\theta,\omega)$ is negative. Assume thus that $i_t(\omega) > 0$. Defining then

$$\alpha_t(\theta,\omega) := \begin{cases} -i_t(\omega)/s_t(\theta,\omega) & \text{if } s_t(\theta,\omega) < 0 \\ \infty & \text{otherwise,} \end{cases}$$

we have that, in the range $\alpha \in [0,\infty)$, $g_{t,\alpha}(\theta,\omega) \geq 0$ if and only if $\alpha \leq \alpha_t(\theta,\omega)$ (if $\alpha_t(\theta,\omega) = \infty$, then this is obviously to be read as $g_{t,\alpha}(\theta,\omega) \geq 0$ for any $\alpha \in [0,\infty)$).

Assume for a moment that

$$\min\big\{\alpha_t(\theta,\omega) : (\theta,\omega) \in [-\pi,\pi]\times[0,\pi]\big\} = \begin{cases} \tilde\alpha_{t,\rho} (< 1) & \text{for } t \in \mathcal{D}_\rho^{\text{cv}} \\ 1 & \text{for } t \in \mathcal{E}_\rho. \end{cases} \tag{S.3.7}$$

Then, for any $t \in \mathcal{D}_\rho^{\text{cv}}$ and $(\theta,\omega) \in [-\pi,\pi]\times[0,\pi]$, we have $\tilde\alpha_\rho \leq \tilde\alpha_{t,\rho} \leq \alpha_t(\theta,\omega)$, so that $g_{t,\tilde\alpha_\rho}(\theta,\omega) \geq 0$. This establishes (i)$_a$ above. Similarly, for any $t \in \mathcal{E}_\rho$ and $(\theta,\omega) \in [-\pi,\pi]\times[0,\pi]$, we then have $\tilde\alpha_\rho \leq 1 \leq \alpha_t(\theta,\omega)$, so that $g_{t,\tilde\alpha_\rho}(\theta,\omega) \geq 0$, which proves (i)$_b$. Finally, for any $\alpha > \tilde\alpha_\rho$, there exists $t \in \mathcal{D}_\rho^{\text{cv}}$ such that $\alpha > \tilde\alpha_{t,\rho}$, which, according to (S.3.7), implies that there exists (θ,ω) such that $\alpha > \alpha_t(\theta,\omega)$. With these t, θ and ω, we then have $g_{t,\alpha}(\theta,\omega) < 0$, which proves (ii) above. Therefore, it only remains to establish (S.3.7).

To do so, fix $t \in \mathcal{D}_\rho^{\text{cv}} \cup \mathcal{E}_\rho$ arbitrarily. For any fixed $\omega \in [0,\pi]$, the fact that $i_t(\omega)$ is positive and does not depend on θ implies that

$$\alpha_t(\omega) := \min\{\alpha_t(\theta,\omega) : \theta \in [-\pi,\pi]\} = -\frac{i_t(\omega)}{\min\{s_t(\theta,\omega) : \theta \in [-\pi,\pi]\}}.$$

Since we safely excluded the case for which $q_t = 0$ and $(\cos\omega)^2 = 1$, we have $1 + (q_t - 1)(\cos\omega)^2 > 0$, so that the Cauchy–Schwarz inequality readily yields

$$\begin{aligned}\alpha_t(\omega) &= \frac{i_t(\omega)}{(p_t-1)\sqrt{(1+(q_t-1)(\cos\omega)^2)^2 + (\sin(2\omega))^2}} \\ &= \frac{p_t + (p_t q_t - p_t - q_t)(\cos\omega)^2}{(p_t-1)\sqrt{(q_t-3)(q_t+1)(\cos\omega)^4 + 2(q_t+1)(\cos\omega)^2 + 1}}.\end{aligned}$$

Obviously, $\inf\{\alpha_t(\omega) : \omega \in [0,\pi]\} = \inf\{f_t(\lambda) : \lambda \in [0,1]\}$, with

$$f_t(\lambda) = \frac{p_t + (p_t q_t - p_t - q_t)\lambda}{(p_t-1)\sqrt{(q_t-3)(q_t+1)\lambda^2 + 2(q_t+1)\lambda + 1}}.$$

A direct calculation shows that

$$f_t'(\lambda) = \frac{(2p_t - q_t)(q_t+1)\lambda - (2p_t+q_t)}{(p_t-1)((q_t-3)(q_t+1)\lambda^2 + 2(q_t+1)\lambda+1)^{3/2}}. \tag{S.3.8}$$

We need to consider two cases. (a) $t \in \mathcal{D}_\rho^{\rm cv}$. Provided that $2p_t - q_t \neq 0$, f_t admits a single critical point, namely

$$\lambda_{t_*} := \frac{2p_t+q_t}{(2p_t-q_t)(q_t+1)}.$$

It is easy to check that if $2p_t - q_t - 2 > 0$, then $\lambda_{t_*} \in (0,1)$. Writing $u(t) = t^2/\rho(t)$, a direct computation shows that

$$2p_t - q_t - 2 = \frac{t^2 u''(t)}{u(t) - tu'(t)} = \frac{(\rho(t))^2 u''(t)}{t\psi_-(t) - \rho(t)}. \tag{S.3.9}$$

Since $t \in \mathcal{D}_\rho^{\rm cv}$, this yields $2p_t - q_t - 2 > 0$. Therefore, λ_{t_*} is well-defined and $\lambda_{t_*} \in (0,1)$. Clearly, $f'(\lambda) < 0$ for $\lambda \in [0, \lambda_{t_*})$ and $f'(\lambda) > 0$ for $\lambda \in (\lambda_{t_*}, 1]$, so that

$$\min\big\{\alpha_t(\theta,\omega) : (\theta,\omega) \in [-\pi,\pi]\times[0,\pi]\big\} = \min\{f_t(\lambda) : \lambda \in [0,1]\} = f_t(\lambda_{t_*}),$$

which, after easy computations, provides

$$\min\big\{\alpha_t(\theta,\omega) : (\theta,\omega) \in [-\pi,\pi]\times[0,\pi]\big\} = \tilde{\alpha}_{t,\rho},$$

as was to be showed in (S.3.7); note that $\tilde{\alpha}_{t,\rho} = f_t(\lambda_{t_*}) < f_t(1) = 1$.

(b) $t \in \mathcal{E}_\rho$. For such a t, (S.3.9) entails that $2p_t - q_t - 2 \leq 0$. For any $\lambda \in (0,1)$, we thus have

$$\begin{aligned}\ell_t(\lambda) &:= (2p_t - q_t)(q_t+1)\lambda - (2p_t+q_t)\\ &\leq \max(\ell_t(0), \ell_t(1)) = \max(-(2p_t+q_t), (2p_t - q_t - 2)q_t) \leq 0.\end{aligned}$$

It then follows from (S.3.8) that f_t is monotone non-increasing in $[0,1]$, so that its minimal value over $[0,1]$ is $f_t(1) = 1$. Consequently,

$$\min\big\{\alpha_t(\theta,\omega) : (\theta,\omega) \in [-\pi,\pi]\times[0,\pi]\big\} = 1.$$

This ends the proof of (S.3.7), hence establishes the theorem in the case where ρ is twice continuously differentiable on $(0,\infty)$.

Extension to the general case is then straightforward. To be more specific, assume now that ρ is only piecewise twice continuously differentiable. If $\alpha \leq \tilde{\alpha}_\rho$, then we proved above that the Hessian matrix $\nabla^2 H^\rho_{\alpha,u}(x)$ is positive semi-definite for any $u \in \mathcal{S}^{d-1}$ and $x \in \mathbb{R}^d$ such that $\|x\| = t \in \mathcal{D}_\rho$. Proceeding exactly as in the second part of the proof of Theorem 3.1 (with $H^\rho_{\alpha,u}$ instead of $H^\rho_{1,u}$), the convexity of ρ then implies that $x \mapsto H^\rho_{\alpha,u}(x)$ is convex over $\mathbb{R}^d$ for any $u \in \mathcal{S}^{d-1}$, which shows that $\alpha_\rho \geq \tilde{\alpha}_\rho$. Finally, if $\alpha > \tilde{\alpha}_\rho$, then the first part of the proof shows that there exist $t \in \mathcal{D}_\rho$ and $u, v, y \in \mathcal{S}^{d-1}$ such that $v'\nabla^2 H^\rho_{\alpha,u}(ty)v < 0$, which of course implies that $x \mapsto H^\rho_{\alpha,u}(x)$ is not convex over $\mathbb{R}^d$. Therefore, $\alpha_\rho \leq \tilde{\alpha}_\rho$, which establishes the result. □

Corollary S.3.1. *Let $\rho \in \mathcal{C}$ be such that $t \mapsto \rho(t)/t$ is convex on $(0, \infty)$. Then $\alpha_\rho \geq \sqrt{2/3} \approx .8165$.*

Proof of Corollary S.3.1. Recall from (S.3.9) that, for any $t \in \mathcal{D}^{\text{cv}}_\rho$, we have $2p_t - q_t - 2 > 0$, that is, $p_t > (q_t + 2)/2$. Since, for any $q > 0$, the map

$$p \mapsto \sqrt{\frac{q(4p^2 - 4p - q)}{4(p-1)^2(q+1)}}$$

is monotone non-increasing in $((q + 2)/2, \infty)$, Theorem 3.2 yields

$$\alpha_\rho \geq \inf_{t \in \mathcal{D}^{\text{cv}}_\rho} \lim_{p \to \infty} \sqrt{\frac{q_t(4p^2 - 4p - q_t)}{4(p-1)^2(q_t+1)}} = \inf_{t \in \mathcal{D}^{\text{cv}}_\rho} \sqrt{\frac{q_t}{q_t + 1}}.$$

The result then follows from the fact that the convexity of $t \mapsto v(t) = \rho(t)/t$ entails that $q_t = 2 + (tv''(t)/v'(t)) \geq 2$ for any $t \in \mathcal{D}^{\text{cv}}_\rho$, where v' and v'' stand for the first and second derivatives of v, respectively (these are well-defined on $\mathcal{D}_\rho$). □

The proof of Theorem 3.3 requires the following lemma.

Lemma S.3.3. *Let $\rho \in \mathcal{C}$. Fix $x \in \mathbb{R}^d \setminus \{0\}$ and $y \in \mathbb{R}^d$ such that $\rho(\|x\|) + \rho(\|y\|) - 2\rho(\|(x + y)/2\|) = 0$. Then, there exists $\lambda \geq 0$ such that $y = \lambda x$.*

Proof of Lemma S.3.3. Since ρ is monotone strictly increasing on $[0, \infty)$ (Lemma S.1.4) and convex, we have $2\rho(\|x + y\|/2) \leq 2\rho(\|x\|/2 + \|y\|/2) \leq \rho(\|x\|) + \rho(\|y\|)$. The assumption on x and y entails that these inequalities must be equalities. Since ρ is monotone strictly increasing, we must then have $\|x + y\| = \|x\| + \|y\|$, so that $y = \lambda x$ for some $\lambda \geq 0$. □

Proof of Theorem 3.3. (i) Fix $\alpha \in [0, \alpha_\rho) \cup \{0\}$. Since the map $\mu \mapsto M^\rho_{\alpha,u}(\mu)$ is continuous (Lemma S.2.1), it is enough to show that it is midpoint strictly convex. Assume ad absurdum that there exist $\mu_1, \mu_2 \in \mathbb{R}^d$, with $\mu_1 \neq \mu_2$, such that $M^\rho_{\alpha,u}(\mu_1) + M^\rho_{\alpha,u}(\mu_2) - 2M^\rho_{\alpha,u}((\mu_1 + \mu_2)/2) \leq 0$. Since convexity of $H^\rho_{\alpha,u}$ (which holds since $\alpha \leq \alpha_\rho$)

trivially implies convexity of $M^{\rho}_{\alpha,u}$, we must then have $M^{\rho}_{\alpha,u}(\mu_1)+M^{\rho}_{\alpha,u}(\mu_2)-2M^{\rho}_{\alpha,u}((\mu_1+\mu_2)/2)=0$, that is

$$\int_{\mathbb{R}^d}\left\{H^{\rho}_{\alpha,u}(z-\mu_1)+H^{\rho}_{\alpha,u}(z-\mu_2)-2H^{\rho}_{\alpha,u}(z-(\mu_1+\mu_2)/2)\right\}dP(z)=0.$$

The convexity of $H^{\rho}_{\alpha,u}$ implies that

$$H^{\rho}_{\alpha,u}(z-\mu_1)+H^{\rho}_{\alpha,u}(z-\mu_2)-2H^{\rho}_{\alpha,u}(z-(\mu_1+\mu_2)/2)=0 \tag{S.3.10}$$

for P-almost all z. Now, fix z satisfying (S.3.10). Using the notation introduced in the proof of Lemma 3.1, we then have $a(z-\mu_1,z-\mu_2)+\alpha u'V(z-\mu_1,z-\mu_2)=0$. If $\alpha=0$, then we have $a(z-\mu_1,z-\mu_2)=0$. If $\alpha\in(0,\alpha_\rho)$, then

$$\begin{aligned} 0 &= a(z-\mu_1,z-\mu_2)+\alpha u'V(z-\mu_1,z-\mu_2)\\ &\geq a(z-\mu_1,z-\mu_2)-\alpha\|V(z-\mu_1,z-\mu_2)\|\\ &\geq a(z-\mu_1,z-\mu_2)-\alpha_\rho\|V(z-\mu_1,z-\mu_2)\|\geq 0, \end{aligned}$$

since $\rho\in\mathcal{C}_{\alpha_\rho}=V_{\alpha_\rho}$ by definition. These inequalities must therefore be equalities, so that $V(z-\mu_1,z-\mu_2)=0$ (since $\alpha<\alpha_\rho$), which in turn implies that $a(z-\mu_1,z-\mu_2)=0$. For any $\alpha\in[0,\alpha_\rho)\cup\{0\}$, we thus obtained that $a(z-\mu_1,z-\mu_2)=0$, that is,

$$\rho(\|z-\mu_1\|)+\rho(\|z-\mu_2\|)-2\rho(\|z-(\mu_1+\mu_2)/2\|). \tag{S.3.11}$$

Since $\mu_1\neq\mu_2$, we cannot have that both $z-\mu_1$ and $z-\mu_2$ are equal to the zero vector in $\mathbb{R}^d$. Without any loss of generality, assume that $z-\mu_1\neq 0$. Lemma S.3.3 then implies that $z-\mu_2=\lambda(z-\mu_1)$ for some $\lambda\in[0,\infty)\setminus\{1\}$ (since $\mu_1\neq\mu_2$, we cannot have $\lambda=1$), so that, in particular, z belongs to the line containing μ_1 and μ_2.

For (S.3.10) to be satisfied for P-almost all z, we must thus have that P is concentrated on the line containing μ_1 and μ_2. Now, note that, with $f(t)=\rho(t\|z-\mu_1\|)$, it follows from (S.3.11) that

$$f(1)+f(\lambda)-2f((1+\lambda)/2)=0.$$

Since f is convex on $[0,\infty)$, it follows that f is an affine function on the open interval with endpoints λ and 1, which in turn implies that ρ is an affine function on the open interval with endpoints $\lambda\|z-\mu_1\|$ and $\|z-\mu_1\|$. Consequently, there exists an open interval on which ψ_- is constant. Summing up, we showed that P is concentrated on a line and that there exists an open interval on which ψ_- is constant. Since this is a contradiction, we conclude that $M^{\rho}_{\alpha,u}$ is midpoint strictly convex, hence strictly convex.

Now, for $\alpha\in[0,\alpha_\rho)\cup\{0\}$, it follows from Theorem 2.1 that at least one ρ-quantile $\mu^{\rho}_{\alpha,u}$—that is, a global minimizer of $\mu\mapsto M^{\rho}_{\alpha,u}(\mu)$—exists. Strict convexity of $M^{\rho}_{\alpha,u}$ of course implies that this minimizer is unique. □

S.4. Proofs for Section 4

The proof of Theorem 4.1 requires the following lemma.

Lemma S.4.1. *Let $P \in \mathcal{P}_d^\rho$ be spherically symmetric about the origin of $\mathbb{R}^d$ and let $\mu \in \mathbb{R}^d \setminus \{0\}$. Then, $P[\{z \in \mathbb{R}^d \setminus \{0\} : \|z-\mu\| \notin \mathcal{D}_\rho\}] = 0$.*

PROOF OF LEMMA S.4.1. First note that $\{z \in \mathbb{R}^d \setminus \{0\} : \|z-\mu\| \notin \mathcal{D}_\rho\} = \{\mu\} \cup B$, where

$$B := \bigcup_{t \in (0,\infty)\setminus \mathcal{D}_\rho} B_t$$

involves $B_t := \{z \in \mathbb{R}^d \setminus \{0\} : \|z-\mu\| = t\}$. The sphericity assumption implies that $P[\{\mu\}] = 0$. Since, by assumption, $(0,\infty) \setminus \mathcal{D}_\rho$ is at most countable, it is then sufficient to prove that $P[B_t] = 0$ for any $t \in (0,\infty) \setminus \mathcal{D}_\rho$.

To do so, fix $t \in (0,\infty) \setminus \mathcal{D}_\rho$. Pick arbitrarily $v \in \mathcal{S}^{d-1}$ with $v'\mu = 0$, and partition B_t into $B_t = B_{t,\geq} \cup B_{t,<}$, with $B_{t,\geq} := \{z \in B_t : v'(z-\mu) \geq 0\}$ and $B_{t,<} := \{z \in B_t : v'(z-\mu) < 0\}$. Sphericity implies that $P[B_{t,\geq}] \geq P[B_{t,<}]$, so that $P[B_t] = P[B_{t,\geq}] + P[B_{t,<}] \leq 2P[B_{t,\geq}]$. Let then O_k, $k = 1, 2, \ldots$, be pairwise different $d \times d$ orthogonal matrices fixing the orthogonal complement to $\{\lambda\mu + \eta v : \lambda, \eta \in \mathbb{R}\}$. By construction, the sets $O_k B_{t,\geq} = \{O_k z : z \in B_{t,\geq}\}$, $k = 1, 2, \ldots$, are pairwise disjoint. Since sphericity implies that $P[O_k B_{t,\geq}] = P[B_{t,\geq}]$ for any k, we must then have $P[B_{t,\geq}] = 0$. It follows that $P[B_t] \leq 2P[B_{t,\geq}] = 0$, which establishes the result. □

PROOF OF THEOREM 4.1. (i) Fix $\alpha = 0$ and $u \in \mathcal{S}^{d-1}$, and assume ad absurdum that $\mu \neq 0$ is a minimizer of $M^\rho_{\alpha,u}$. Then, the sphericity assumption implies that $-\mu$ is a minimizer, too (Proposition 2.1). It thus follows from Theorem 3.3 that P is concentrated on a line. From sphericity, we must then have P is a Dirac at the origin of $\mathbb{R}^d$, which provides $M^\rho_{\alpha,u}(\mu) = \rho(\|\mu\|)$ for any $\mu \in \mathbb{R}^d$. Since $\rho(0) = 0$ and $\rho(t) > 0$ for any $t > 0$, we conclude that the only minimizer of $M^\rho_{\alpha,u}$ is the origin of $\mathbb{R}^d$, a contradiction.

(ii) Fix $\alpha > 0$ and $u \in \mathcal{S}^{d-1}$. We consider two cases.

(A) Fix an arbitrary μ that does not belong to the line $\{\lambda u : \lambda \in \mathbb{R}\}$. Then $t := \|\mu\| > 0$ and $w := \mu/\|\mu\| \in \mathcal{S}^{d-1} \setminus \{u\}$. Since $\mu = tw \neq 0$, the sphericity assumption implies that $P[\{\mu\}]=0$. Letting $v=(I-ww')u=u-(u'w)w$, Proposition 5.1 then yields

$$\frac{\partial M^{\rho}_{\alpha,u}}{\partial v}(\mu)=-\alpha v'\mathrm{E}\bigg[\frac{\rho(\|Z_\mu\|)}{\|Z_\mu\|}\xi_{Z,\mu}\bigg]u+\alpha v'\mathrm{E}\bigg[\frac{\rho(\|Z_\mu\|)}{\|Z_\mu\|}\frac{Z_\mu Z_\mu'}{\|Z_\mu\|^2}\xi_{Z,\mu}\bigg]u$$

$$-v'\mathrm{E}\bigg[\{\psi_-(\|Z_\mu\|)\mathbb{I}[v'Z_\mu>0]+\psi_+(\|Z_\mu\|)\mathbb{I}[v'Z_\mu<0]\}\bigg(1+\alpha\frac{u'Z_\mu}{\|Z_\mu\|}\bigg)\frac{Z_\mu}{\|Z_\mu\|}\bigg]$$

$$=:-\alpha\mathrm{E}\bigg[\frac{\rho(\|Z_\mu\|)}{\|Z_\mu\|}\xi_{Z,\mu}\bigg](1-(u'w)^2)+T_1+T_2, \tag{S.4.12}$$

say. Since $v'\mu=0$, we have

$$\begin{aligned}T_1 &= \alpha\mathrm{E}\bigg[\frac{\rho(\|Z_\mu\|)}{\|Z_\mu\|}\frac{(v'Z)Z_\mu'(v+(u'w)w)}{\|Z_\mu\|^2}\xi_{Z,\mu}\bigg]\\ &= \alpha\mathrm{E}\bigg[\frac{\rho(\|Z_\mu\|)}{\|Z_\mu\|}\frac{(v'Z)^2}{\|Z_\mu\|^2}\xi_{Z,\mu}\bigg]+\alpha(u'w)\mathrm{E}\bigg[\frac{\rho(\|Z_\mu\|)}{\|Z_\mu\|}\frac{(v'Z)(w'Z-t)}{\|Z_\mu\|^2}\xi_{Z,\mu}\bigg].\end{aligned}$$

Pick a $d\times d$ orthogonal matrix O such that $Ow=w$ and $Ov=-v$ (such a matrix O exists since w and v are orthogonal). Since OZ and Z are equal in distribution and since $\|OZ-\mu\|=\|O(Z-\mu)\|=\|Z_\mu\|$ almost surely, we have

$$\mathrm{E}\bigg[\frac{\rho(\|Z_\mu\|)}{\|Z_\mu\|}\frac{(v'Z)(w'Z-t)}{\|Z_\mu\|^2}\xi_{Z,\mu}\bigg]=-\mathrm{E}\bigg[\frac{\rho(\|Z_\mu\|)}{\|Z_\mu\|}\frac{(v'Z)(w'Z-t)}{\|Z_\mu\|^2}\xi_{Z,\mu}\bigg]=0,$$

so that

$$T_1=\alpha\mathrm{E}\bigg[\frac{\rho(\|Z_\mu\|)}{\|Z_\mu\|}\frac{(v'Z)^2}{\|Z_\mu\|^2}\xi_{Z,\mu}\bigg]. \tag{S.4.13}$$

Now, turning to T_2, note that multiplying the random variable in the expectation by the indicator $\mathbb{I}[Z\in A]$, with $A=\{z\in\mathbb{R}^d:\|z-\mu\|\in\mathcal{D}_\rho\}$ will not change the value of T_2. Indeed, this only discards, in the corresponding integral in z, (a) the value $z=0$, that makes the integrand equal to zero (recall that $v'\mu=0$) and (b) the non-zero values of z such that $\|z-\mu\|\notin\mathcal{D}_\rho$, that form a set with P-probability zero (Lemma S.4.1). Consequently,

$$\begin{aligned}T_2 &= -v'\mathrm{E}\bigg[\psi_-(\|Z_\mu\|)\bigg(1+\alpha\frac{u'Z_\mu}{\|Z_\mu\|}\bigg)\frac{Z_\mu}{\|Z_\mu\|}\xi_{Z,\mu}\bigg]\\ &= -\mathrm{E}\bigg[\psi_-(\|Z_\mu\|)\frac{v'Z}{\|Z_\mu\|}\xi_{Z,\mu}\bigg]-\alpha\mathrm{E}\bigg[\psi_-(\|Z_\mu\|)\frac{(v'Z)(u'Z_\mu)}{\|Z_\mu\|^2}\xi_{Z,\mu}\bigg]\\ &= -\mathrm{E}\bigg[\psi_-(\|Z_\mu\|)\frac{v'Z}{\|Z_\mu\|}\xi_{Z,\mu}\bigg]-\alpha\mathrm{E}\bigg[\psi_-(\|Z_\mu\|)\frac{(v'Z)^2}{\|Z_\mu\|^2}\xi_{Z,\mu}\bigg]\\ &\qquad -\alpha(u'w)\mathrm{E}\bigg[\psi_-(\|Z_\mu\|)\frac{(v'Z)(w'Z-t)}{\|Z_\mu\|^2}\xi_{Z,\mu}\bigg].\end{aligned}$$

Using again the fact that OZ and Z are equal in distribution and that $\|OZ-\mu\| = \|Z_\mu\|$ almost surely, this provides

$$T_2 = -\alpha \mathrm{E}\Big[\psi_-(\|Z_\mu\|)\frac{(v'Z)^2}{\|Z_\mu\|^2}\xi_{Z,\mu}\Big].$$

Jointly with (S.4.13), this shows that

$$T_1 + T_2 = -\alpha \mathrm{E}\bigg[\bigg(\psi_-(\|Z_\mu\|) - \frac{\rho(\|Z_\mu\|)}{\|Z_\mu\|}\bigg)\frac{(v'Z)^2}{\|Z_\mu\|^2}\xi_{Z,\mu}\bigg] \leq 0,$$

since $\psi_-(t) \geq \rho(t)/t$ for any $t > 0$. Therefore, (S.4.12) provides

$$\frac{\partial M^\rho_{\alpha,u}}{\partial v}(\mu) \leq -\alpha \mathrm{E}\Big[\frac{\rho(\|Z_\mu\|)}{\|Z_\mu\|}\xi_{Z,\mu}\Big](1-(u'w)^2) < 0,$$

which shows that μ is not a minimizer of $M^\rho_{\alpha,u}$.

(B) Fix $\mu = -tu$, with $t > 0$. Since we still have $P[\{\mu\}] = 0$, Proposition 5.1 provides

$$\frac{\partial M^\rho_{\alpha,u}}{\partial u}(\mu) = -\alpha \mathrm{E}\Big[\frac{\rho(\|Z_\mu\|)}{\|Z_\mu\|}\xi_{Z,\mu}\Big] + \alpha \mathrm{E}\Big[\frac{\rho(\|Z_\mu\|)}{\|Z_\mu\|}\frac{(u'Z_\mu)^2}{\|Z_\mu\|^2}\xi_{Z,\mu}\Big]$$
$$-\mathrm{E}\bigg[\{\psi_-(\|Z_\mu\|)\mathbb{I}[u'Z_\mu > 0] + \psi_+(\|Z_\mu\|)\mathbb{I}[u'Z_\mu < 0]\}\bigg(1 + \alpha\frac{u'Z_\mu}{\|Z_\mu\|}\bigg)\frac{u'Z_\mu}{\|Z_\mu\|}\bigg].$$

From Lemma S.4.1, we obtain

$$\frac{\partial M^\rho_{\alpha,u}}{\partial u}(\mu) = -\alpha \mathrm{E}\Big[\frac{\rho(\|Z_\mu\|)}{\|Z_\mu\|}\xi_{Z,\mu}\Big] + \alpha \mathrm{E}\Big[\frac{\rho(\|Z_\mu\|)}{\|Z_\mu\|}\frac{(u'Z_\mu)^2}{\|Z_\mu\|^2}\xi_{Z,\mu}\Big]u$$
$$-\mathrm{E}\bigg[\psi_-(\|Z_\mu\|)\bigg(1 + \alpha\frac{u'Z_\mu}{\|Z_\mu\|}\bigg)\frac{u'Z_\mu}{\|Z_\mu\|}\xi_{Z,\mu}\mathbb{I}[Z \neq 0]\mathbb{I}[\|Z-\mu\| \in \mathcal{D}_\rho]\bigg] - \psi_-(t)(1+\alpha)P[\{0\}],$$

which rewrites

$$\frac{\partial M^\rho_{\alpha,u}}{\partial u}(\mu) = -\alpha \mathrm{E}\Big[\frac{\rho(\|Z_\mu\|)}{\|Z_\mu\|}\xi_{Z,\mu}\Big] - \mathrm{E}\Big[\psi_-(\|Z_\mu\|)\frac{u'Z_\mu}{\|Z_\mu\|}\xi_{Z,\mu}\Big]$$
$$-\alpha \mathrm{E}\bigg[\bigg(\psi_-(\|Z_\mu\|) - \frac{\rho(\|Z_\mu\|)}{\|Z_\mu\|}\bigg)\frac{(u'Z_\mu)^2}{\|Z_\mu\|^2}\xi_{Z,\mu}\bigg].$$

Since $\psi(t) \geq \rho(t)/t > 0$ for any $t > 0$, this yields

$$\frac{\partial M^\rho_{\alpha,u}}{\partial u}(\mu) < -\mathrm{E}\Big[\psi_-(\|Z_\mu\|)\frac{u'Z_\mu}{\|Z_\mu\|}\xi_{Z,\mu}\Big] = -\mathrm{E}\Big[\psi_-(\|Z_\mu\|)\frac{u'Z+t}{\|Z_\mu\|}\xi_{Z,\mu}\Big]. \tag{S.4.14}$$

Now, let Γ_u be an arbitrary $d \times (d-1)$ matrix whose columns form an orthonormal basis of the orthogonal complement to u in $\mathbb{R}^d$, and define the random $(d-1)$-vector Y

through $Z = (u'Z)u + \Gamma_u Y$. Note that $\|Z_\mu\|^2 = \|Z + tu\|^2 = \|(u'Z + t)u + \Gamma_u Y\|^2 = (u'Z+t)^2+\|Y\|^2$ and that, for any $r \geq 0$, the distribution of $u'Z$, conditional on $\|Y\| = r$, is symmetric about zero. Therefore, the monotonicity of ψ_- implies that

$$\mathrm{E}\bigg[\frac{\psi_-(((u'Z+t)^2+\|Y\|^2)^{1/2})(u'Z+t)}{((u'Z+t)^2+\|Y\|^2)^{1/2}}\xi_{Z,\mu}\mathbb{I}[|u'Z|>t]\,\bigg|\,\|Y\|=r\bigg] \geq 0$$

for any $r \geq 0$. It follows that

$$\mathrm{E}\bigg[\psi_-(\|Z_\mu\|)\frac{u'Z+t}{\|Z_\mu\|}\xi_{Z,\mu}\bigg] \geq \mathrm{E}\bigg[\psi_-(\|Z_\mu\|)\frac{u'Z+t}{\|Z_\mu\|}\xi_{Z,\mu}\mathbb{I}[|u'Z|>t]\bigg]$$

$$= \mathrm{E}\bigg[\mathrm{E}\bigg[\frac{\psi_-(((u'Z+t)^2+\|Y\|^2)^{1/2})(u'Z+t)}{((u'Z+t)^2+\|Y\|^2)^{1/2}}\xi_{Z,\mu}\mathbb{I}[|u'Z|>t]\bigg|\,\|Y\|\bigg]\bigg] \geq 0.$$

We conclude that the partial derivative in (S.4.14) is strictly negative, hence that μ is not a minimizer of $M^\rho_{\alpha,u}$. Together with the result proved in case (A), this establishes that any minimizer of $M^\rho_{\alpha,u}$ belongs to the halfline $\{\lambda u : \lambda \geq 0\}$. □

Proposition S.4.1. *Let $\rho \in \mathcal{C}$ and $P \in \mathcal{P}^\rho_d$. Fix $u \in \mathcal{S}^{d-1}$. Assume that P is spherically symmetric about the origin of $\mathbb{R}^d$. Then, (i) for $\rho(t) = t$, the origin of $\mathbb{R}^d$ is the ρ-quantile of P of order α in direction u if and only if $\alpha \leq P[\{0\}]$ (in which case this quantile is unique); (ii) if $\psi_+(0)P[\{0\}] + P[\|Z\| \in (0,\infty) \setminus \mathcal{D}_\rho] = 0$, then, for $\alpha \in (0,1)$, any ρ-quantile $\mu^\rho_{\alpha,u}$ belongs to the halfline $\{\lambda u : \lambda > 0\}$.*

Proof of Proposition S.4.1. (i) First note that, for $\rho(t) = t$, it readily follows from Proposition 5.1 that

$$\frac{\partial M^\rho_{\alpha,u}}{\partial u}(0) = (1-\alpha)P[\{0\}] - \alpha\mathrm{E}[\xi_{Z,0}] - \mathrm{E}\bigg[\frac{u'Z}{\|Z\|}\xi_{Z,0}\bigg] = P[\{0\}] - \alpha. \qquad \text{(S.4.15)}$$

We then consider three cases. (a) For $\alpha = 0$, the origin of $\mathbb{R}^d$ is the only ρ-quantile of order α in direction u (Theorem 4.1(i)). (b) For $\alpha \in (0, P[\{0\}]]$, any ρ-quantile of order α in direction u belongs to $\{\lambda u : \lambda \geq 0\}$ (Theorem 4.1(ii)), and the directional derivative in (S.4.15) is non-negative. Convexity of $\mu \mapsto M^\rho_{\alpha,u}$ implies that $(\partial M^\rho_{\alpha,u}/\partial u)(tu)$ is a monotone non-increasing function of t, that will thus take non-negative values for any $t > 0$. This implies that the origin of $\mathbb{R}^d$ is a ρ-quantile of order α in direction u. Ad absurdum, assume then that this ρ-quantile is not unique. Then Theorem 3.3 implies that P is concentrated on a line, hence that $P[\{0\}] = 1$. It results that $M^\rho_{\alpha,u} = \|\mu\| - au'\mu$, which is minimized at $\mu = 0$ only, a contradiction. (c) For $\alpha \in (P[\{0\}], 1)$, the directional derivative in (S.4.15) is strictly negative, so that the origin of $\mathbb{R}^d$ is not a ρ-quantile of order α in direction u.

(ii) Since $\psi_+(0)P[\{0\}] + P[\|Z\| \in (0,\infty) \setminus \mathcal{D}_\rho] = 0$, Proposition 5.1 provides

$$\begin{aligned}\frac{\partial M^\rho_{\alpha,u}}{\partial u}(0) &= -\alpha \mathrm{E}\Big[\frac{\rho(\|Z\|)}{\|Z\|}\xi_{Z,0}\Big] + \alpha \mathrm{E}\Big[\frac{\rho(\|Z\|)}{\|Z\|}\frac{(u'Z)^2}{\|Z\|^2}\xi_{Z,0}\Big] \\ &\qquad - \mathrm{E}\Big[\psi_-(\|Z\|)\Big(1+\alpha\frac{u'Z}{\|Z\|}\Big)\frac{u'Z}{\|Z\|}\xi_{Z,0}\Big] \\ &= -\alpha \mathrm{E}\Big[\frac{\rho(\|Z\|)}{\|Z\|}\xi_{Z,0}\Big] - \alpha \mathrm{E}\Big[\Big(\psi_-(\|Z\|) - \frac{\rho(\|Z\|)}{\|Z\|}\Big)\frac{(u'Z)^2}{\|Z\|^2}\xi_{Z,0}\Big] < 0,\end{aligned}$$

which shows that $\mu = 0$ cannot be a minimizer of $M^\rho_{\alpha,u}$. The result thus follows from Theorem 4.1(ii). □

Proof of Theorem 4.2. By definition, $\mathcal{C}^{\rm sph}_0 \subseteq \mathcal{C}$, whereas Lemma 3.1 provides $\mathcal{C} = \mathcal{C}_0 \subseteq \mathcal{C}^{\rm sph}_0$. Therefore, $\mathcal{C}^{\rm sph}_0 = \mathcal{C}$. For any $\alpha \in [0,1]$, let

$$V^{\rm sph}_\alpha := \big\{\rho \in \mathcal{C} : g^\rho_{\alpha,u}(z-su, z-tu) \geq 0 \ \forall z \in \mathbb{R}^d \ \forall s,t \in \mathbb{R} \ \forall u \in \mathcal{S}^{d-1}\big\},$$

where $g^\rho_{\alpha,u}(x,y) := a(x,y) - \alpha|u'V(x,y)|$ is based on the same quantities $a(x,y)$ and $V(x,y)$ as in the proof of Lemma 3.1. Note that

$$\begin{aligned}&H^\rho_{\alpha,u}(z-su) + H^\rho_{\alpha,u}(z-tu) - 2H^\rho_{\alpha,u}(z-(s+t)u/2) \\ &\qquad = a(z-su, z-tu) + \alpha u'V(z-su, z-tu).\end{aligned} \tag{S.4.16}$$

Obviously, if $\rho \in V^{\rm sph}_\alpha$, then (S.4.16) implies that $t \mapsto H^\rho_{\alpha,u}(z-tu)$ is midpoint convex for any $u \in \mathcal{S}^{d-1}$, that is, $\rho \in \mathcal{C}^{\rm sph}_\alpha$. Conversely, if $\rho \in \mathcal{C}^{\rm sph}_\alpha$, then, writing $c_0 = -\mathrm{Sign}(u'V(z-su, z-tu))c$ for any quantity c, we have

$$\begin{aligned}&a(z-su, z-tu) - \alpha|u'V(z-su, z-tu)| \\ &\qquad = a(z-s_0u_0, z-t_0u_0) + \alpha u_0'V(z-s_0u_0, z-t_0u_0) \geq 0,\end{aligned}$$

so that $\rho \in V^{\rm sph}_\alpha$. Therefore, $\mathcal{C}^{\rm sph}_\alpha = V^{\rm sph}_\alpha$, which implies that $\mathcal{C}^{\rm sph}_{\alpha_2} = V^{\rm sph}_{\alpha_2} \subseteq V^{\rm sph}_{\alpha_1} = \mathcal{C}^{\rm sph}_{\alpha_1}$ for any $\alpha_1 < \alpha_2$.

Note that if $\alpha^{\rm sph}_\rho = 0$, then we need to have $\alpha = 0$ and the result follows from Theorem 3.3. We may thus assume that $\alpha^{\rm sph}_\rho > 0$. Fix then $\alpha \in [0, \alpha^{\rm sph}_\rho)$ and $u \in \mathcal{S}^{d-1}$. We proceed as in the proof of Theorem 3.3. From continuity of $t \mapsto M^\rho_{\alpha,u}(tu)$ (Lemma S.2.1), it is sufficient to prove that this map is midpoint strictly convex. Assume ad absurdum that there exist $s,t > 0$, with $s \neq t$, such that $M^\rho_{\alpha,u}(su) + M^\rho_{\alpha,u}(tu) - 2M^\rho_{\alpha,u}((s+t)u/2) \leq 0$, that is,

$$\int_{\mathbb{R}^d} \big\{H^\rho_{\alpha,u}(z-su) + H^\rho_{\alpha,u}(z-tu) - 2H^\rho_{\alpha,u}(z-(s+t)u/2)\big\}\, dP(z) \leq 0.$$

Since $\alpha \leq \alpha_\rho^{\rm sph}$, the integrand is non-negative for any z, so that the integral must be equal to zero, which (using again the fact that the integrand is non-negative for any z) entails that

$$H^\rho_{\alpha,u}(z-su)+H^\rho_{\alpha,u}(z-tu)-2H^\rho_{\alpha,u}(z-(s+t)u/2)=0 \tag{S.4.17}$$

for P-almost all z. Any z satisfying (S.4.17) satisfies

$$\begin{aligned} 0 &= a(z-su,z-tu)+\alpha u'V(z-su,z-tu) \\ &\geq a(z-su,z-tu)-\alpha|u'V(z-su,z-tu)| \\ &\geq a(z-su,z-tu)-\alpha_\rho^{\rm sph}|u'V(z-su,z-tu)| \geq 0, \end{aligned}$$

since $\rho \in \mathcal{C}_{\alpha_\rho^{\rm sph}} = V_{\alpha_\rho^{\rm sph}}$ by definition. Since $\alpha < \alpha_\rho^{\rm sph}$, we must have $u'V(z-su,z-tu)=0$, hence also $a(z-su,z-tu)=0$, for P-almost all z. Thus,

$$\rho(\|z-su\|)+\rho(\|z-tu\|)-2\rho(\|z-(s+t)u/2\|)$$

for P-almost all z, which, in view of Lemma S.3.3, entails that P is concentrated on the line $\{\lambda u : \lambda \in \mathbb{R}\}$. The sphericity assumption then implies that $P[\{0\}]=1$, which yields

$$M^\rho_{\alpha,u}(tu)=\rho(|t|)\left(1-\alpha\frac{t}{|t|}\right)\xi_{t,0}=\rho(|t|)\{(1+\alpha)\mathbb{I}[t<0]+(1-\alpha)\mathbb{I}[t\geq 0]\}.$$

Thus, $t \mapsto M^\rho_{\alpha,u}(tu)$ is strictly convex on $\mathbb{R}$, so that $M^\rho_{\alpha,u}(su)+M^\rho_{\alpha,u}(tu)-2M^\rho_{\alpha,u}((s+t)u/2)>0$, a contradiction. We thus proved that $t \mapsto M^\rho_{\alpha,u}(tu)$ is always strictly convex on $(0,\infty)$ under sphericity, which, in view of Theorem 4.1, implies uniqueness of $\mu^\rho_{\alpha,u}$. □

Proof of Theorem 4.3. We prove the result only in the case where ρ is twice continuously differentiable on $(0,\infty)$ (extension to the general case where ρ is piecewise twice continuously differentiable on $(0,\infty)$ can indeed be done as in the proof of Theorem 3.2). Letting $\tilde{\alpha}^{\rm sph}_{t,\rho} := \sqrt{\beta_{p_t,q_t}}$ for any $t \in \mathcal{D}^{\rm sph}_\rho$, we need to prove that

$$\alpha_\rho^{\rm sph}=\tilde{\alpha}_\rho^{\rm sph} := \begin{cases} \inf_{t\in\mathcal{D}_\rho^{\rm sph}} \tilde{\alpha}^{\rm sph}_{t,\rho} & \text{if } \mathcal{D}_\rho^{\rm sph}\neq\emptyset \\ 1 & \text{otherwise} \end{cases} \tag{S.4.18}$$

and that $\tilde{\alpha}_\rho^{\rm sph}<1$ if $\mathcal{D}_\rho^{\rm sph}\neq\emptyset$ (we will actually show below that $\tilde{\alpha}^{\rm sph}_{t,\rho}<1$ for any $t \in \mathcal{D}_\rho^{\rm sph}$). To prove that $\alpha_\rho^{\rm sph}=\tilde{\alpha}_\rho^{\rm sph}$, we need to show that (i) for any $\alpha \leq \tilde{\alpha}_\rho^{\rm sph}$,

$$\left(\frac{d^2}{dr^2}H^\rho_{\alpha,u}(z-ru)\big|_{r=s}=\right) u'\nabla^2 H^\rho_{\alpha,u}(z-su)u \geq 0$$

for any $u \in \mathcal{S}^{d-1}$, $r > 0$ and $z \in \mathbb{R}^d$ such that $z - su \neq 0$ and that (ii) if $\tilde{\alpha}_\rho^{\rm sph} < 1$, then, for any $\alpha > \tilde{\alpha}_\rho^{\rm sph}$, there exist such values of z, s, u providing $u'\nabla^2 H^\rho_{\alpha,u}(z - su)u < 0$. Clearly, letting $x = z - su$ and writing $x = ty$, with $t = \|x\|$ and $y = x/\|x\|$, we have to show that (i) $u'\nabla^2 H^\rho_{\alpha,u}(ty)u \geq 0$ for any $t > 0$, $u, y \in \mathcal{S}^{d-1}$ and $\alpha \leq \tilde{\alpha}_\rho^{\rm sph}$ and that (ii) if $\tilde{\alpha}_\rho^{\rm sph} < 1$, then, for any $\alpha > \tilde{\alpha}_\rho^{\rm sph}$, there exist such values of t, u, y for which $u'\nabla^2 H^\rho_{\alpha,u}(ty)u < 0$.

To prove (i)–(ii), partition $\mathcal{D}_\rho = (0, \infty)$ into $\mathcal{D}_\rho^{\rm cv} \cup \mathcal{E}_\rho \cup \mathcal{F}_\rho$ as in the proof of Theorem 3.2. It follows from (S.3.6) that if $t \in \mathcal{F}_\rho$, then $u'\nabla^2 H^\rho_{\alpha,u}(ty)u$ for any $\alpha \in [0, 1]$ and $u, y \in \mathcal{S}^{d-1}$. Now, recall from the proof of Theorem 3.2 that for $t \in (\mathcal{D}_\rho \setminus \mathcal{F}_\rho) = \mathcal{D}_\rho^{\rm cv} \cup \mathcal{E}_\rho$,

$$\frac{t^2}{\rho(t)} v'\nabla^2 H^\rho_{\alpha,u}(ty)v = g_{t,\alpha}(\theta, \omega),$$

with

$$g_{t,\alpha}(\theta, \omega) := (p_t - 1)(q_t - 2)(1 + \alpha \cos\theta)(\cos\omega)^2 + 1 - (\cos\omega)^2$$
$$+ (p_t - 1)\big\{(1 + \alpha\cos\theta)(1 - (\cos\omega)^2) + 2(\cos\omega)^2 + 2\alpha(\cos\omega)\cos(\theta - \omega)\big\},$$

where $\omega = \arccos(v'y) \in [0, \pi]$ is the angle between v and y and $\theta = s_{u,v,y} \arccos(u'y) \in [-\pi, \pi]$ is the signed angle between u and y. A close inspection of the proof of Theorem 3.2 reveals that we established there that if $t \in \mathcal{E}_\rho$, then $g_{t,\alpha}(\theta, \omega) \geq 0$ for any $\alpha \in [0, 1]$ and (θ, ω), which ensures that $u'\nabla^2 H^\rho_{\alpha,u}(ty)u$ for any $t \in \mathcal{E}_\rho$, $\alpha \in [0, 1]$ and $u, y \in \mathcal{S}^{d-1}$.

Therefore, it remains to prove that (i) $u'\nabla^2 H^\rho_{\alpha,u}(ty)u \geq 0$ for any $t \in \mathcal{D}_\rho^{\rm cv}$, $u, y \in \mathcal{S}^{d-1}$ and $\alpha \leq \tilde{\alpha}_\rho^{\rm sph}$, and that (ii) if $\tilde{\alpha}_\rho^{\rm sph} < 1$, then, for any $\alpha > \tilde{\alpha}_\rho^{\rm sph}$, there exist $t \in \mathcal{D}_\rho^{\rm cv}$ and $u, y \in \mathcal{S}^{d-1}$ such that $u'\nabla^2 H^\rho_{\alpha,u}(ty)u < 0$. To do so, put, for any $\omega \in [0, \pi]$,

$$h_{t,\alpha}(\omega) := g_{t,\alpha}(\omega, \omega) = \frac{t^2}{\rho(t)} u'\nabla^2 H^\rho_{\alpha,u}(ty)u.$$

Following the same approach as in the proof of Theorem 3.2, write then

$$h_{t,\alpha}(\omega) = i_t(\omega) + \alpha s_t^{\rm sph}(\omega),$$

where the intercept function $i_t(\omega)$ is still given by

$$i_t(\omega) = p_t + (p_t q_t - p_t - q_t)(\cos\omega)^2$$

and where the slope function $s_t^{\rm sph}(\omega)$ is now defined as

$$s_t^{\rm sph}(\omega) := (p_t - 1)(\cos\omega)\{(q_t - 3)(\cos\omega)^2 + 3\}.$$

Since the intercept is the same as in the proof of Theorem 3.2, we still have that $i_t(\omega) \geq 0$ and that if $i_t(\omega) = 0$, then $q_t = 0$ and $(\cos\omega)^2 = 1$, so that $h_{t,\alpha}(\omega) = 0$ for any $\alpha \in [0, 1]$

and $\omega \in [0,\pi]$. From now one, we thus safely restrict the case where $i_t(\omega) > 0$ when investigating when $h_{t,\alpha}(\omega)$ is negative. Putting then

$$\alpha_t^{\rm sph}(\omega) := \begin{cases} -i_t(\omega)/s_t^{\rm sph}(\omega) & \text{if } s_t^{\rm sph}(\omega) < 0 \\ \infty & \text{otherwise,} \end{cases}$$

we have that, in the range $\alpha \in [0,\infty)$, $h_{t,\alpha}(\omega) \geq 0$ if and only if $\alpha \leq \alpha_t^{\rm sph}(\omega)$ (if $\alpha_t^{\rm sph}(\omega) = \infty$, then this is to be read as $h_{t,\alpha}(\omega) \geq 0$ for any $\alpha \in [0,\infty)$).

In the beginning of the proof, we defined $\tilde{\alpha}_{t,\rho}^{\rm sph} := \sqrt{\beta_{p_t,q_t}}$ for any $t \in \mathcal{D}_\rho^{\rm sph}$. Let further $\tilde{\alpha}_{t,\rho}^{\rm sph} := 1$ for any $t \in \mathcal{D}_\rho^{\rm cv} \setminus \mathcal{D}_\rho^{\rm sph}$. To establish the theorem, it is then sufficient to prove that

$$\min\left\{\alpha_t^{\rm sph}(\omega) : \omega \in [0,\pi]\right\} = \tilde{\alpha}_{t,\rho}^{\rm sph} \qquad \text{for any } t \in \mathcal{D}_\rho^{\rm cv}, \tag{S.4.19}$$

$$\tilde{\alpha}_{t,\rho}^{\rm sph} < 1 \text{ for any } t \in \mathcal{D}_\rho^{\rm sph} \tag{S.4.20}$$

and

$$\mathcal{D}_\rho^{\rm sph} \subseteq \mathcal{D}_\rho^{\rm cv}. \tag{S.4.21}$$

Indeed, for any $\alpha \leq \tilde{\alpha}_\rho^{\rm sph}$, $t \in \mathcal{D}_\rho^{\rm cv}$ and $\omega \in [0,\pi]$, we then have $\alpha \leq \tilde{\alpha}_{t,\rho}^{\rm sph} \leq \alpha_t^{\rm sph}(\omega)$, hence $h_{t,\alpha}(\omega) \geq 0$. This proves (i) (note that if $\mathcal{D}_\rho^{\rm cv}$ is empty, then Theorem 3.2 yields $\alpha_\rho = 1$, which, since $\alpha_\rho^{\rm sph} \geq \alpha_\rho$, implies that $\alpha_\rho^{\rm sph} = 1$, as claimed by the theorem since (S.4.21) entails that $\mathcal{D}_\rho^{\rm sph}$ is then empty, too). Now, assume that $\tilde{\alpha}_\rho^{\rm sph} < 1$. Then, for any $\alpha > \tilde{\alpha}_\rho^{\rm sph}$, there exists $t \in \mathcal{D}_\rho^{\rm cv}$ such that $\alpha > \tilde{\alpha}_{t,\rho}^{\rm sph}$, which, in view of (S.4.19), ensures that $h_{t,\alpha}(\omega) < 0$ for some $\omega \in [0,\pi]$. This establishes (ii), hence the result.

It thus remains to prove (S.4.19)–(S.4.21). Since $\alpha_t^{\rm sph}(\omega)$ depends on ω only through $\cos\omega$ and since nonnegative values of $\cos\omega$ provide $\alpha_t^{\rm sph}(\omega) = \infty$, one has

$$\min\left\{\alpha_t^{\rm sph}(\omega) : \omega \in [0,\pi]\right\} = \min\left\{\ell_t(s) : s \in (0,1]\right\} =: m_t,$$

where we let

$$\ell_t(s) := \frac{p_t + (p_t q_t - p_t - q_t)s}{(p_t - 1)\sqrt{s}((q_t - 3)s + 3)}.$$

Since $\ell_t(s)$ diverges to infinity as $s \to 0$ from above, the minimum of ℓ_t in $s \in (0,1]$ indeed exists, and it is equal to the minimum between $\ell_t(1) = 1$ and the infimum of ℓ_t over $(0,1)$. The derivative of ℓ_t at $s \in (0,1)$ has the same sign as

$$d_t(s) := a_t s^2 + 3(2p_t - q_t)s - 3p_t, \quad \text{with } a_t := (3 - q_t)(p_t q_t - p_t - q_t).$$

Below, we use the term “root” for a value of s such that $d_t(s) = 0$. Obviously, there are at most two roots in $(0,1)$ and $d_t(0) = -3p_t < 0$. In the rest of the proof, we organize the discussion in two levels, the first one ((A)–(B)) involving the sign of

$$d_t(1) = q_t(3(p_t - 2) - (p_t q_t - p_t - q_t))$$

and the second one ((1)–(3)) associated with the sign of a_t.

(A) Assume that $d_t(1) > 0$. Since $q_t \geq 0$, we then have $3(p_t - 2) - (p_t q_t - p_t - q_t) > 0$, which rewrites $4p_t + q_t - p_t q_t - 6 > 0$, i.e., $t \in \mathcal{D}_\rho^{\mathrm{sph}}$. Note that $3(p_t - 2) - (p_t q_t - p_t - q_t) > 0$ entails that $q_t < (2(2p_t - 3))/(p_t - 1) \leq 2(p_t - 1)$, so that $2p_t - q_t - 2 > 0$. In view of (S.3.9), this proves (S.4.21).

Since $d_t(0) < 0$ and d_t is convex/concave, there is exactly one root in $(0, 1)$, that is the minimizer of ℓ_t over $(0, 1]$ (since $d_t(s) < 0$ below this root and $d_t(s) > 0$ above it). If (A1) $a_t > 0$, then $s \mapsto d_t(s)$ is convex, so that the root in $(0, 1)$ is the larger root, namely

$$r_t = \frac{-3(2p_t - q_t) + \sqrt{\Delta_t}}{2a_t}, \tag{S.4.22}$$

where

$$\begin{aligned} \Delta_t &= 9(2p_t - q_t)^2 + 12p_t a_t \\ &= 12q_t\Big(p_t - \frac{3}{2}\Big)\Big(\frac{3}{2}(2p_t - q_t) - (p_t q_t - p_t - q_t)\Big). \end{aligned}$$

In this case, the minimum is thus $m_t = \ell_t(r_t)(< \ell_t(1) = 1)$, which, after some computations, is shown to be equal to $\tilde{\alpha}_{t,\rho}^{\mathrm{sph}}$ (recall that $t \in \mathcal{D}_\rho^{\mathrm{sph}}$). If (A2) $a_t < 0$, then $s \mapsto d_t(s)$ is concave, so that the root in $(0, 1)$ is now the smaller root, which is still r_t. The minimum is thus $m_t = \ell_t(r_t) = \tilde{\alpha}_{t,\rho}^{\mathrm{sph}}$, too. If (A3) $a_t = 0$, then either $q_t = 3$ or $p_t q_t - p_t - q_t = 0$. If $q_t = 3$, then $p_t > 3$ (since $d_t(1) > 0$) and the root (in $(0, 1)$) is $p_t/(2p_t - 3)$, which is the limit of r_t as $q_t \to 3$. Thus, the minimum in this case is still $m_t = \ell_t(r_t) = \tilde{\alpha}_{t,\rho}^{\mathrm{sph}}$ (recall from the statement of the theorem that the value $\tilde{\alpha}_{t,\rho}^{\mathrm{sph}} = \sqrt{\beta_{p_t, q_t}}$ is defined as a limit when q_t makes its value undefined). If $p_t q_t - p_t - q_t = 0$, then $p_t > 2$ (since $d_t(1) > 0$) and the root in $(0, 1)$ is $(p_t - 1)/(2p_t - 3)$, which is the limit of r_t as $q_t \to p_t/(p_t - 1)$. Thus, the minimum in this case is once more $m_t = \ell_t(r_t) = \tilde{\alpha}_{t,\rho}^{\mathrm{sph}}$.

(B) Assume that $d_t(1) \leq 0$, so that $t \in \mathcal{D}_\rho^{\mathrm{cv}} \setminus \mathcal{D}_\rho^{\mathrm{sph}}$. If (B1) $a_t > 0$, then convexity of $s \mapsto d_t(s)$ implies that $d_t(s) \leq 0$ for any $s \in [0, 1]$ (recall that $d_t(0) < 0$), so that the minimum is $m_t = \ell_t(1) = 1 = \tilde{\alpha}_{t,\rho}^{\mathrm{sph}}$. If (B2) $a_t < 0$, then $s \mapsto d_t(s)$ is concave, and there might have zero, one or two roots in $(0, 1]$. If there is zero, one root, or two roots with the smaller root—namely, r_t in (S.4.22)—larger than or equal to one, then $d_t(s) \leq 0$ for any $s \in (0, 1]$, so that the minimum is $m_t = \ell_t(1) = 1 = \tilde{\alpha}_{t,\rho}^{\mathrm{sph}}$. Assume then that there are two roots and that $m_t < 1$. Both roots are positive (since their sum and products are positive), and they must both belong to $(0, 1]$ (since $r_t < 1$ and $d_t(1) \leq 0$). Thus we must have $d_t'(1) < 0$, which yields

$$p_t q_t - p_t - q_t > 3(p_t - \tfrac{3}{2}). \tag{S.4.23}$$

We need to consider three cases: (a) $p_t > \frac{3}{2}$. Then, $p_t q_t - p_t - q_t > 0$, so that $q_t > 3$ (recall that $a_t < 0$). Thus, (S.4.23) yields $\frac{3}{2}(2p_t - q_t) - (p_t q_t - p_t - q_t) \geq 3(3-q)/2 < 0$, which implies that $\Delta_t \leq 0$, a contradiction since we assumed that there are two roots. (b) $p_t \leq \frac{3}{2}$ and $p_t q_t - p_t - q_t \leq 0$. Since $2p_t - q_t \geq 2$ (recall that $t \in \mathcal{D}_\rho^{\rm cv}$), we then have $\Delta_t \leq 0$, which provides the same contradiction as above. (c) $p_t \leq \frac{3}{2}$ and $p_t q_t - p_t - q_t > 0$. Since $a_t < 0$, we have $q_t > 3 \geq 2p_t$. Therefore, $2p_t - q_t - 2 < 0$, which contradicts the fact that $t \in \mathcal{D}_\rho^{\rm cv}$. Finally, (B3) if $a_t = 0$, then $s \mapsto d_t(s)$ is linear and both $d_t(0)$ and $d_t(1)$ are nonpositive. Thus, $d_t(s) \leq 0$ for any $s \in (0,1]$, so that the minimum is still $m_t = \ell_t(1) = 1 = \tilde{\alpha}_{t,\rho}^{\rm sph}$.

Let us summarize the findings of this discussion. Any $t \in \mathcal{D}_\rho^{\rm sph}$ corresponds to case (A), where the minimum $\min\{\alpha_t^{\rm sph}(\omega) : \omega \in [0,\pi]\} = \tilde{\alpha}_{t,\rho}^{\rm sph}(< 1)$, whereas any $t \in \mathcal{D}_\rho^{\rm cv} \setminus \mathcal{D}_\rho^{\rm sph}$ corresponds to case (B), where the minimum $\min\{\alpha_t^{\rm sph}(\omega) : \omega \in [0,\pi]\} = \tilde{\alpha}_{t,\rho}^{\rm sph} = 1$. This proves both (S.4.19) and (S.4.20). Since (S.4.21) was established when discussing case (A), the result is proved. □

S.5. Proofs for Section 5

In this section, for a map $g : \mathbb{R}^k \to \mathbb{R}$ admitting directional derivatives at $x(\in \mathbb{R}^k)$ in any direction $v(\in \mathbb{R}^d \setminus \{0\})$, we write $\nabla g(x)$ for the vector stacking the k partial derivatives $\partial_{x_\ell} g(x)$, $\ell = 1, \ldots, k$, on top of each other, irrespective of whether or not g is differentiable at x.

Lemma S.5.1. *Let $\rho \in \mathcal{C}$, $\alpha \in [0,1]$ and $u \in \mathcal{S}^{d-1}$. Then, (i) for any $x \in \mathbb{R}^d$ and $v \in \mathbb{R}^d \setminus \{0\}$, $H_{\alpha,u}^\rho$ admits a directional derivative at x in direction v, given by*

$$\frac{\partial H_{\alpha,u}^\rho}{\partial v}(x) = \Big(\psi_-(\|x\|)\mathbb{I}[v'x < 0] + \psi_+(\|x\|)\mathbb{I}[v'x > 0]\Big)\left(1 + \alpha\frac{u'x}{\|x\|}\right)\frac{v'x}{\|x\|}\xi_{x,0}$$
$$+ \alpha\frac{\rho(\|x\|)}{\|x\|}v'\left(u - \frac{(u'x)x}{\|x\|^2}\right)\xi_{x,0} + \psi_+(0)(\|v\| + \alpha u'v)\mathbb{I}[x = 0].$$

(ii) If $x_0 \in \mathbb{R}^d$ is such that $\|x_0\| \in \mathcal{D}_\rho$, then $H_{\alpha,u}^\rho$ is twice continuously differentiable in a neighbourhood $\mathcal{N}$ of x_0, with gradient

$$\nabla H_{\alpha,u}^\rho(x) = \psi_-(\|x\|)\left(1 + \alpha\frac{u'x}{\|x\|}\right)\frac{x}{\|x\|} + \alpha\frac{\rho(\|x\|)}{\|x\|}\left(u - \frac{(u'x)x}{\|x\|^2}\right)$$

and Hessian matrix

$$\nabla^2 H^\rho_{\alpha,u}(x) = \frac{\|x\|^2\psi'_-(\|x\|) - 2\|x\|\psi_-(\|x\|) + 2\rho(\|x\|)}{\|x\|^2}\left(1+\alpha\frac{u'x}{\|x\|}\right)\frac{xx'}{\|x\|^2}$$
$$+\frac{\rho(\|x\|)}{\|x\|^2}\left(I_d - \frac{xx'}{\|x\|^2}\right)$$
$$+\frac{\|x\|\psi_-(\|x\|) - \rho(\|x\|)}{\|x\|^2}\left\{\left(1+\alpha\frac{u'x}{\|x\|}\right)\left(I_d - \frac{xx'}{\|x\|^2}\right)+2\frac{xx'}{\|x\|^2}+\alpha\frac{xu'+ux'}{\|x\|}\right\}$$

for any $x \in \mathcal{N}$.

Proof of Lemma S.5.1. (i) The proof, which is a direct computation of the directional derivative, is left to the reader. (ii) Fix $x_0 \in \mathbb{R}^d$ such that $\|x_0\| \in \mathcal{D}_\rho$, and write $\varphi(s) = \rho(s)/s$ for $s \in (0,\infty)$. Since ρ (hence, also φ) is twice differentiable over a neighbourhood of $\|x_0\|$, the map $x \mapsto H^\rho_{\alpha,u}(x) = \varphi(\|x\|)(\|x\| + \alpha u'x)$ is twice differentiable over a neighbourhood $\mathcal{N}$ of x_0 and we have

$$\frac{\partial H^\rho_{\alpha,u}}{\partial x_i}(x) = \varphi'(\|x\|)\frac{x_i}{\|x\|}(\|x\| + \alpha u'x) + \varphi(\|x\|)\left(\frac{x_i}{\|x\|} + \alpha u_i\right) \tag{S.5.24}$$

for any $x \in \mathcal{N}$. Leibniz's rule then yields

$$\frac{\partial}{\partial x_j}\left(\frac{\partial H^\rho_{\alpha,u}}{\partial x_i}\right)(x)$$
$$= \left(\varphi''(\|x\|)\frac{x_j x_i}{\|x\|^2} + \frac{\varphi'(\|x\|)}{\|x\|}\delta_{ij} - \varphi'(\|x\|)\frac{x_i x_j}{\|x\|^3}\right)(\|x\| + \alpha u'x)$$
$$+\varphi'(\|x\|)\frac{x_i}{\|x\|}\left(\frac{x_j}{\|x\|} + \alpha u_j\right) + \varphi'(\|x\|)\frac{x_j}{\|x\|}\left(\frac{x_i}{\|x\|} + \alpha u_i\right)$$
$$+\varphi(\|x\|)\left(\delta_{ij}\frac{1}{\|x\|} - \frac{x_i x_j}{\|x\|^3}\right).$$

With $y = x/\|x\|$ and $t = \|x\|$, this rewrites

$$\frac{\partial}{\partial x_j}\left(\frac{\partial H^\rho_{\alpha,u}}{\partial x_i}\right)(x) = t\varphi''(t)(1+\alpha u'y)y_i y_j + \frac{\varphi(t)}{t}(\delta_{ij} - y_i y_j)$$
$$+\varphi'(t)\big\{\delta_{ij}(1+\alpha u'y) - y_i y_j(1+\alpha u'y) + 2y_i y_j + \alpha y_i u_j + \alpha u_i y_j\big\}. \tag{S.5.25}$$

Clearly, (S.5.24) and (S.5.25) yield

$$\begin{aligned}\nabla H^\rho_{\alpha,u}(x) &= t\varphi'(t)(1+\alpha u'y)y + \varphi(t)(y+\alpha u)\\ &= (t\varphi'(t) + \varphi(t))(1+\alpha u'y)y + \alpha\varphi(t)(u - (u'y)y)\end{aligned}$$

and

$$\nabla^2 H^\rho_{\alpha,u}(x) = t\varphi''(t)(1+\alpha u'y)yy' + \frac{\varphi(t)}{t}(I_d - yy')$$
$$+\varphi'(t)\big\{(1+\alpha u'y)(I_d - yy') + 2yy' + \alpha yu' + \alpha uy'\big\},$$

respectively. Expressing φ, φ' and φ'' in terms of ρ provides the result. □

Lemma S.5.2. *Let $\rho \in \mathcal{C}$, $\alpha \in [0,1]$ and $u \in \mathcal{S}^{d-1}$. Then,*

$$\big|H^\rho_{\alpha,u}(z-\mu_2) - H^\rho_{\alpha,u}(z-\mu_1)\big|\xi_{z,\mu_1}\xi_{z,\mu_2}$$
$$\leq (1+3\alpha)\psi_-(\|z-\mu_1\| + \|\mu_2-\mu_1\|)\|\mu_2-\mu_1\|$$

for any $z, \mu_1, \mu_2 \in \mathbb{R}^d$ with $\mu_1 \neq \mu_2$.

Proof of Lemma S.5.2. Since the inequality to be proved trivially follows from Lemma S.1.4(iii) for $z \in \{\mu_1, \mu_2\}$, we restrict to $z \notin \{\mu_1, \mu_2\}$. Write then

$$\big(H^\rho_{\alpha,u}(z-\mu_2) - H^\rho_{\alpha,u}(z-\mu_1)\big)\xi_{z,\mu_1}\xi_{z,\mu_2} = f_{\alpha,u}(z,\mu_1,\mu_2) + g_{\alpha,u}(z,\mu_1,\mu_2),$$

with

$$f_{\alpha,u}(z,\mu_1,\mu_2) := \big(\rho(\|z-\mu_2\|) - \rho(\|z-\mu_1\|)\big)\left(1 + \alpha u' \frac{z-\mu_2}{\|z-\mu_2\|}\right)$$

and

$$g_{\alpha,u}(z,\mu_1,\mu_2) := \alpha\rho(\|z-\mu_1\|)\left(u'\frac{z-\mu_2}{\|z-\mu_2\|} - u'\frac{z-\mu_1}{\|z-\mu_1\|}\right).$$

Lemma S.1.3 and the monotonicity of ψ_- ensure that, for some c between $\|z-\mu_1\|$ and $\|z-\mu_2\|$,

$$|\rho(\|z-\mu_2\|) - \rho(\|z-\mu_1\|)| \leq \psi_-(c)\big|\|z-\mu_2\| - \|z-\mu_1\|\big|$$
$$\leq \psi_-(\|z-\mu_1\| + \|\mu_2-\mu_1\|)\|\mu_2-\mu_1\|,$$

so that

$$|f_{\alpha,u}(z,\mu_1,\mu_2)| \leq (1+\alpha)\psi_-(\|z-\mu_1\| + \|\mu_2-\mu_1\|)\|\mu_2-\mu_1\|.$$

Now, using Lemma S.1.2, Lemma S.1.4(i), and the monotonicity of ψ_- again, provides

$$|g_{\alpha,u}(z,\mu_1,\mu_2)| \leq \alpha\rho(\|z-\mu_1\|)\left\|\frac{z-\mu_2}{\|z-\mu_2\|} - \frac{z-\mu_1}{\|z-\mu_1\|}\right\|$$
$$\leq 2\alpha\frac{\rho(\|z-\mu_1\|)\|\mu_2-\mu_1\|}{\|z-\mu_1\|} \leq (2\alpha\psi_-(\|z-\mu_1\| + \|\mu_2-\mu_1\|)\|\mu_2-\mu_1\|.$$

The result follows. □

PROOF OF PROPOSITION 5.1. (i) For any $t > 0$, write

$$\frac{M^\rho_{\alpha,u}(\mu+tv) - M^\rho_{\alpha,u}(\mu)}{t} = \frac{1}{t}\int_{\mathbb{R}^d} \Big\{ H^\rho_{\alpha,u}(z-\mu-tv) - H^\rho_{\alpha,u}(z-\mu) \Big\}\, dP(z)$$

$$= \frac{\rho(t\|v\|)}{t}\Big(1-\alpha u'\frac{v}{\|v\|}\Big) P[\{\mu\}] - \frac{\rho(t\|v\|)}{t}\Big(1+\alpha u'\frac{v}{\|v\|}\Big) P[\{\mu+tv\}]$$

$$+ \int_{\mathbb{R}^d} \frac{H^\rho_{\alpha,u}(z-\mu-tv) - H^\rho_{\alpha,u}(z-\mu)}{t} \xi_{z,\mu+tv}\xi_{z,\mu}\, dP(z).$$

Since Lemma S.1.4(ii) implies that $\rho(t\|v\|)/t = \|v\|\rho(t\|v\|)/(t\|v\|) \leq \|v\|\rho(\|v\|)/\|v\| = \rho(\|v\|)$ for $t \in (0,1]$, Lemma S.1.1 yields

$$\lim_{t \overset{>}{\to} 0} \frac{\rho(t\|v\|)}{t}\Big(1+\alpha u'\frac{v}{\|v\|}\Big) P[\{\mu+tv\}] = 0.$$

Now, with $\delta = \delta_\mu$ as in (4), Lemma S.5.2 implies that

$$\left| \frac{H^\rho_{\alpha,u}(z-\mu-tv) - H^\rho_{\alpha,u}(z-\mu)}{t} \right| \xi_{z,\mu}\xi_{z,\mu+tv} \leq \|v\|(1+3\alpha)\psi_-(\|z-\mu\|+\delta_\mu)$$

as soon as $t < \delta_\mu/\|v\|$. Applying Lebesgue's DCT, Lemma S.5.1 then provides

$$\lim_{t \overset{>}{\to} 0} \frac{M^\rho_{\alpha,u}(\mu+tv) - M^\rho_{\alpha,u}(\mu)}{t}$$

$$= \psi_+(0)\big(\|v\| - \alpha u'v\big) P[\{\mu\}] + \int_{\mathbb{R}^d} \frac{\partial H^\rho_{\alpha,u}}{\partial(-v)}(z-\mu)\xi_{z,\mu}\, dP(z), \tag{S.5.26}$$

which establishes the result. □

PROOF OF THEOREM 5.1. (i) We start with necessity. Assume thus that the map $M^\rho_{\alpha,u}$ is differentiable at μ_0. Then, directional derivatives at μ_0 in direction v are linear in v, which imposes that

$$\frac{\partial M^\rho_{\alpha,u}}{\partial v}(\mu_0) + \frac{\partial M^\rho_{\alpha,u}}{\partial(-v)}(\mu_0) = 0,$$

that is,

$$\int_{\mathbb{R}^d} \big(\psi_+(\|z-\mu_0\|) - \psi_-(\|z-\mu_0\|)\big) \frac{|v'(z-\mu_0)|}{\|z-\mu_0\|}\Big(1+\alpha\frac{u'(z-\mu_0)}{\|z-\mu_0\|}\Big)\xi_{z,\mu_0} dP(z)$$

$$+2\psi_+(0)P[\{\mu_0\}] = 0$$

for any $v \in \mathcal{S}^{d-1}$. Since convexity of ρ implies that $\psi_+(t) \geq \psi_-(t)$ for any $t > 0$, we must then have that $\psi_+(0)P[\{\mu_0\}] = 0$ and that, for any $v \in \mathcal{S}^{d-1}$,

$$\big(\psi_+(\|z-\mu_0\|) - \psi_-(\|z-\mu_0\|)\big)\frac{|v'(z-\mu_0)|}{\|z-\mu_0\|}\xi_{z,\mu_0} = 0$$

for P-almost every $z \in \mathbb{R}^d$. This implies

$$\begin{aligned} 0 &= \int_{\mathbb{R}^d} \big(\psi_+(\|z-\mu_0\|) - \psi_-(\|z-\mu_0\|)\big)\frac{|v'(z-\mu_0)|}{\|z-\mu_0\|}\xi_{z,\mu_0}\,dP(z) \\ &= \sum_{t\in(0,\infty)\setminus\mathcal{D}_\rho} \frac{1}{t}\big(\psi_+(t) - \psi_-(t)\big)\int_{\mathbb{R}^d} |v'(z-\mu_0)|\mathbb{I}[\|z-\mu_0\| = t]\,dP(z). \end{aligned}$$

This implies that for any $t \in (0,\infty)\setminus\mathcal{D}_\rho$ and $v \in \mathcal{S}^{d-1}$, we have $|v'(z-\mu_0)|\mathbb{I}[\|z-\mu_0\| = t] = 0$ for P-almost every $z \in \mathbb{R}^d$. Fix then $t \in (0,\infty)\setminus\mathcal{D}_\rho$ and let $\mathcal{V} \subset \mathcal{S}^{d-1}$ be dense in $\mathcal{S}^{d-1}$ and at most countable. For any $v \in \mathcal{V}$, there exists a subset $E_v \subseteq \mathbb{R}^d$ with P-probability one such that $|v'(z-\mu_0)|\mathbb{I}[\|z-\mu_0\| = t] = 0$ for any $z \in E_v$. Since $\mathcal{V}$ is countable, $E := \cap_{v\in\mathcal{V}}E_v$ has probability one, too. Thus, $|v'(z-\mu_0)|\mathbb{I}[\|z-\mu_0\| = t] = 0$ for any $v \in \mathcal{V}$ and $z \in E$. Since $\mathcal{V}$ is dense in $\mathcal{S}^{d-1}$ and $v \mapsto |v'(z-\mu_0)|$ is continuous, we have $|v'(z-\mu_0)|\mathbb{I}[\|z-\mu_0\| = t] = 0$ for any $z \in E$ and any $v \in \mathcal{S}^{d-1}$. Taking the supremum over $v \in \mathcal{S}^{d-1}$ then yields $0 = \|z-\mu_0\|\mathbb{I}[\|z-\mu_0\| = t] = t\mathbb{I}[\|z-\mu_0\| = t]$ for any $z \in E$, hence $P[\|Z-\mu_0\| = t] = 0$. Since this holds for any t in the at most countable set $(0,\infty)\setminus\mathcal{D}_\rho$, we conclude that $P[\|Z-\mu_0\| \in (0,\infty)\setminus\mathcal{D}_\rho] = 0$.

Turning to sufficiency, assume now that $\psi_+(0)P[\{\mu_0\}] + P[\|Z-\mu_0\| \in (0,\infty)\setminus\mathcal{D}_\rho] = 0$. It then directly follows from Proposition 5.1 that $\nabla M^\rho_{\alpha,u}(\mu_0)$ takes the form given in the statement of the theorem (recall that we define the gradient as the vector stacking partial derivatives on top of each other, irrespective of whether or not the function is actually differentiable). Further observe that, since the assumption $\psi_+(0)P[\{\mu_0\}] + P[\|Z-\mu_0\| \in (0,\infty)\setminus\mathcal{D}_\rho] = 0$ implies that, for any $v \in \mathbb{R}^d\setminus\{0\}$,

$$\frac{\partial H^\rho_{\alpha,u}}{\partial(-v)}(z-\mu_0) = -\frac{\partial H^\rho_{\alpha,u}}{\partial v}(z-\mu_0)$$

for P-almost any z, (S.5.26) entails that

$$\nabla M^\rho_{\alpha,u}(\mu_0) = -\int_{\mathbb{R}^d\setminus\{\mu_0\}} \nabla H^\rho_{\alpha,u}(z-\mu_0)\,dP(z). \tag{S.5.27}$$

We want to show that

$$\lim_{\mu\to\mu_0} \frac{M^\rho_{\alpha,u}(\mu) - M^\rho_{\alpha,u}(\mu_0) - (\mu-\mu_0)'\nabla M^\rho_{\alpha,u}(\mu_0)}{\|\mu-\mu_0\|} = 0.$$

Let us then write

$$\frac{M^{\rho}_{\alpha,u}(\mu) - M^{\rho}_{\alpha,u}(\mu_0) - (\mu-\mu_0)'\nabla M^{\rho}_{\alpha,u}(\mu_0)}{\|\mu-\mu_0\|}$$

$$= \int_{\mathbb{R}^d} \frac{H^{\rho}_{\alpha,u}(z-\mu) - H^{\rho}_{\alpha,u}(z-\mu_0) + (\mu-\mu_0)'\nabla H^{\rho}_{\alpha,u}(z-\mu_0)}{\|\mu-\mu_0\|} \xi_{z,\mu}\xi_{z,\mu_0}\, dP(z)$$

$$+\frac{H^{\rho}_{\alpha,u}(\mu_0-\mu)}{\|\mu-\mu_0\|}P[\{\mu_0\}] - \frac{H_{\alpha,u}(\mu-\mu_0)}{\|\mu-\mu_0\|}P[\{\mu\}] + \frac{(\mu-\mu_0)'}{\|\mu-\mu_0\|}\nabla H^{\rho}_{\alpha,u}(\mu-\mu_0)P[\{\mu\}]. \quad \text{(S.5.28)}$$

Note that, as $\mu \to \mu_0$,

$$\begin{aligned} 0 &\leq \frac{H^{\rho}_{\alpha,u}(\mu_0-\mu)}{\|\mu-\mu_0\|}P[\{\mu_0\}] \leq (1+\alpha)\frac{\rho(\|\mu-\mu_0\|)}{\|\mu-\mu_0\|}P[\{\mu_0\}] \\ &\leq (1+\alpha)\psi_-(\|\mu-\mu_0\|)P[\{\mu_0\}] \to (1+\alpha)\psi_+(0)P[\{\mu_0\}] = 0, \end{aligned}$$

so that the second term in (S.5.28) is $o(1)$ as $\mu \to \mu_0$. Proceeding in the same way and using Lemma S.1.1 shows that the third term is also $o(1)$ as $\mu \to \mu_0$. Now, the same holds for the fourth term, since

$$\big\|\nabla H^{\rho}_{\alpha,u}(\mu-\mu_0)\big\| P[\{\mu\}] \leq (1+3\alpha)\psi_+(\|\mu-\mu_0\|)P[\{\mu\}]$$

(see Lemma S.5.1(i)) converges to zero as $\mu \to \mu_0$ (this follows from Lemma S.1.1 and from the monotonicity of ψ_+). Therefore it only remains to show that the first term in (S.5.28) is also $o(1)$ as $\mu \to \mu_0$.

To do so, let $z \in \mathbb{R}^d$ be such that $\|z-\mu_0\| \in \mathcal{D}_\rho$. With $\delta = \delta_{\mu_0}$ as in (4), Lemma S.5.2 ensures that, as soon as $\|\mu-\mu_0\| < \delta$, we have that $|H^{\rho}_{\alpha,u}(z-\mu_0) - H^{\rho}_{\alpha,u}(z-\mu)|\xi_{z,\mu_0}\xi_{z,\mu}/\|\mu-\mu_0\|$ is upper-bounded by the P-integrable function $z \mapsto (1+3\alpha)\psi_-(\|z-\mu_0\|+\delta)$, that does not depend on μ. Moreover, Lemma S.5.1(ii) yields

$$\left|\frac{(\mu-\mu_0)'\nabla H^{\rho}_{\alpha,u}(z-\mu_0)}{\|\mu-\mu_0\|}\right| \leq \|\nabla H^{\rho}_{\alpha,u}(z-\mu_0)\| \leq (1+3\alpha)\psi_-(\|z-\mu_0\|),$$

which is P-integrable and does not depend on μ. Since $P[\|Z-\mu_0\| \in \mathcal{D}_\rho] = 1$ by assumption, we may thus apply Lebesgue's DCT, which, by differentiability of $H^{\rho}_{\alpha,u}$ at $z-\mu_0 (\neq 0)$ (see Lemma S.5.1(ii) again), entails that

$$\lim_{\mu\to\mu_0} \frac{M^{\rho}_{\alpha,u}(\mu_0) - M^{\rho}_{\alpha,u}(\mu) - \nabla M^{\rho}_{\alpha,u}(\mu_0)'(\mu-\mu_0)}{\|\mu-\mu_0\|} = 0.$$

We conclude that $M^{\rho}_{\alpha,u}$ is differentiable at μ_0.

(ii) Let $\mathcal{N}$ be an open subset of $\mathbb{R}^d$ such that $\psi_+(0)P[\{\mu\}]+P[\|Z-\mu\| \in (0,\infty)\backslash\mathcal{D}_\rho] = 0$ for any $\mu \in \mathcal{N}$. Part (i) of the result guarantees that $\mu \mapsto M^\rho_{\alpha,u}(\mu)$ is differentiable at any $\mu \in \mathcal{N}$, with corresponding gradient $\nabla M^\rho_{\alpha,u}(\mu) = v(\mu) - \alpha T(\mu)u$. Using again the inequality $\rho(t)/t \leq \psi_-(t)$ for any $t > 0$, a direct application of Lebesgue's DCT shows that the maps $\mu \mapsto v(\mu)$ and $\mu \mapsto T(\mu)$ are continuous on $\mathcal{N}$, so that the gradient map $\mu \mapsto \nabla M^\rho_{\alpha,u}(\mu)$ is also continuous on $\mathcal{N}$. This establishes continuous differentiability on $\mathcal{N}$. □

The proof of Theorem 5.2 requires the following lemma.

Lemma S.5.3. *Let the assumptions of Theorem 5.2 hold. If Assumption (A) holds, then, for any $\mu \in \mathbb{R}^d$, $v \in \mathbb{R}^d \setminus \{0\}$, $t > 0$, and $z \in \mathbb{R}^d \setminus \{\mu, \mu + tv\}$,*

(i) $$\left|\frac{\rho(\|z-\mu-tv\|)}{\|z-\mu-tv\|} - \frac{\rho(\|z-\mu\|)}{\|z-\mu\|}\right| \leq \frac{\psi_-(\|z-\mu\|)}{\|z-\mu\|}\, t\|v\|$$

and

(ii) $$\big|\psi_-(\|z-\mu-tv\|) - \psi_-(\|z-\mu\|)\big| \leq \frac{\psi_-(\|z-\mu\|)}{\|z-\mu\|}\, t\|v\|,$$

whereas if Assumption (A′) holds, then

(iii) $$\left|\frac{\rho(\|z-\mu-tv\|)}{\|z-\mu-tv\|} - \frac{\rho(\|z-\mu\|)}{\|z-\mu\|}\right| \leq \frac{\psi_-(\|z-\mu\|+r)}{\|z-\mu\|+r}\, t\|v\|$$

for any $\mu \in \mathbb{R}^d$, $v \in \mathbb{R}^d \setminus \{0\}$, $t \in (0, r/\|v\|]$, $z \in \mathbb{R}^d \setminus \{\mu, \mu+tv\}$, and

(iv) $$\big|\psi_-(\|z-\mu-tv\|) - \psi_-(\|z-\mu\|)\big| \leq \psi'_-(\|z-\mu\|+r)t\|v\|$$

for any $\mu \in \mathbb{R}^d$, $v \in \mathbb{R}^d \setminus \{0\}$, $t \in (0, r/\|v\|]$, and $z \in \mathbb{R}^d$ (in (iii)–(iv), r is as in Assumption (A′)).

Proof of Lemma S.5.3. (i) Let $\mu \in \mathbb{R}^d$, $v \in \mathbb{R}^d \setminus \{0\}$, $t > 0$, and $z \in \mathbb{R}^d \setminus \{\mu, \mu+tv\}$. Assume first that $\|z-\mu-tv\| < \|z-\mu\|$. Since $s \mapsto \rho(s)/s$ is monotone non-decreasing on $(0,\infty)$ (Lemma S.1.4), we have

$$\begin{aligned}\left|\frac{\rho(\|z-\mu-tv\|)}{\|z-\mu-tv\|} - \frac{\rho(\|z-\mu\|)}{\|z-\mu\|}\right| &= \frac{\rho(\|z-\mu\|)}{\|z-\mu\|} - \frac{\rho(\|z-\mu-tv\|)}{\|z-\mu-tv\|} \\ &\leq \frac{\rho(\|z-\mu\|)}{\|z-\mu\|} - \frac{\|z-\mu-tv\|}{\|z-\mu\|}\frac{\rho(\|z-\mu-tv\|)}{\|z-\mu-tv\|} \\ &= \frac{\rho(\|z-\mu\|) - \rho(\|z-\mu-tv\|)}{\|z-\mu\|}.\end{aligned}$$

Using Lemma S.1.3 and the fact that ψ_- is monotone non-decreasing, we thus obtain (here, $c \in (\|z-\mu-tv\|, \|z-\mu\|)$)

$$\left|\frac{\rho(\|z-\mu-tv\|)}{\|z-\mu-tv\|} - \frac{\rho(\|z-\mu\|)}{\|z-\mu\|}\right| \leq \frac{\psi_-(c)\big(\|z-\mu\| - \|z-\mu-tv\|\big)}{\|z-\mu\|}$$

$$\leq \frac{\psi_-(\|z-\mu\|)}{\|z-\mu\|} t\|v\|.$$

Assume then that $\|z-\mu-tv\| > \|z-\mu\|$. The same reasoning leads to

$$\left|\frac{\rho(\|z-\mu-tv\|)}{\|z-\mu-tv\|} - \frac{\rho(\|z-\mu\|)}{\|z-\mu\|}\right| = \frac{\rho(\|z-\mu-tv\|)}{\|z-\mu-tv\|} - \frac{\rho(\|z-\mu\|)}{\|z-\mu\|}$$

$$\leq \frac{\rho(\|z-\mu-tv\|)}{\|z-\mu-tv\|} - \frac{\|z-\mu\|}{\|z-\mu-tv\|} \frac{\rho(\|z-\mu\|)}{\|z-\mu\|}$$

$$= \frac{\rho(\|z-\mu-tv\|) - \rho(\|z-\mu\|)}{\|z-\mu-tv\|},$$

which yields (for some $c \in (\|z-\mu\|, \|z-\mu-tv\|)$)

$$\left|\frac{\rho(\|z-\mu-tv\|)}{\|z-\mu-tv\|} - \frac{\rho(\|z-\mu\|)}{\|z-\mu\|}\right| \leq \frac{\psi_-(c)\big(\|z-\mu-tv\| - \|z-\mu\|\big)}{\|z-\mu-tv\|}$$

$$\leq \frac{\psi_-(\|z-\mu-tv\|)}{\|z-\mu-tv\|} t\|v\| \leq \frac{\psi_-(\|z-\mu\|)}{\|z-\mu\|} t\|v\|,$$

where the last inequality follows from the fact that

$$s \mapsto \frac{\psi_-(s)}{s} = \frac{\psi_-(s) - \psi_+(0)}{s-0} + \frac{\psi_+(0)}{s}$$

is, as the sum of two monotone non-increasing functions on $(0, \infty)$ (recall that ψ_- is concave under Assumption (A)), itself monotone non-increasing on $(0, \infty)$. Since the inequality in (i) trivially holds when $\|z-\mu-tv\| = \|z-\mu\|$, the result is proved.

(ii) Let $\mu \in \mathbb{R}^d$, $v \in \mathbb{R}^d \setminus \{0\}$, $t > 0$, and $z \in \mathbb{R}^d \setminus \{\mu, \mu+tv\}$. If $\|z-\mu-tv\| < \|z-\mu\|$, then

$$|\psi_-(\|z-\mu-tv\|) - \psi_-(\|z-\mu\|)| = \psi_-(\|z-\mu\|) - \psi_-(\|z-\mu-tv\|)$$

$$= \frac{\psi_-(\|z-\mu\|) - \psi_-(\|z-\mu-tv\|)}{\|z-\mu\| - \|z-\mu-tv\|}\big(\|z-\mu\| - \|z-\mu-tv\|\big)$$

$$\leq \frac{\psi_-(\|z-\mu\|)}{\|z-\mu\|} t\|v\|,$$

where we used the fact that, since $s \mapsto \psi_-(s)/s$ is monotone non-increasing,

$$\frac{\psi_-(u) - \psi_-(v)}{u - v} \leq \frac{\psi_-(u)}{u}$$

for any $u > v > 0$. If $\|z - \mu - tv\| > \|z - \mu\|$, then the same argument yields

$$\begin{aligned}|\psi_-(\|z-\mu-tv\|) - \psi_-(\|z-\mu\|)| &= \psi_-(\|z-\mu-tv\|) - \psi_-(\|z-\mu\|) \\ &= \frac{\psi_-(\|z-\mu-tv\|) - \psi_-(\|z-\mu\|)}{\|z-\mu-tv\| - \|z-\mu\|}\big(\|z-\mu-tv\| - \|z-\mu\|\big) \\ &\leq \frac{\psi_-(\|z-\mu-tv\|)}{\|z-\mu-tv\|}\, t\|v\| \leq \frac{\psi_-(\|z-\mu\|)}{\|z-\mu\|}\, t\|v\|,\end{aligned}$$

where the last inequality results from the fact that $s \mapsto \psi_-(s)/s$ is monotone non-increasing. Since the inequality in (ii) also trivially holds when $\|z - \mu - tv\| = \|z - \mu\|$, the result is proved.

(iii) Let $\mu \in \mathbb{R}^d$, $v \in \mathbb{R}^d \setminus \{0\}$, $t \in (0, r/\|v\|]$, and $z \in \mathbb{R}^d \setminus \{\mu, \mu + tv\}$. Proceeding as in Part (i) of the proof yields

$$\begin{aligned}&\left|\frac{\rho(\|z-\mu-tv\|)}{\|z-\mu-tv\|} - \frac{\rho(\|z-\mu\|)}{\|z-\mu\|}\right| \\ &\qquad\qquad \leq \max\left(\frac{\psi_-(\|z-\mu\|)}{\|z-\mu\|}, \frac{\psi_-(\|z-\mu-tv\|)}{\|z-\mu-tv\|}\right) t\|v\|.\end{aligned}$$

Since convexity of ψ_- implies that

$$t \mapsto \frac{\psi_-(t)}{t} = \frac{\psi_-(t) - \psi_+(0)}{t - 0}$$

is monotone non-decreasing on $(0, \infty)$, we then obtain

$$\left|\frac{\rho(\|z-\mu-tv\|)}{\|z-\mu-tv\|} - \frac{\rho(\|z-\mu\|)}{\|z-\mu\|}\right| \leq \frac{\psi_-(\|z-\mu\| + r)}{\|z-\mu\| + r}\, t\|v\|.$$

(iv) Let $\mu \in \mathbb{R}^d$, $v \in \mathbb{R}^d \setminus \{0\}$, $t \in (0, r/\|v\|]$, and $z \in \mathbb{R}^d$. Lemma S.1.3 guarantees the existence of a c between $\|z - \mu\|$ and $\|z - \mu - tv\|$ such that

$$\begin{aligned}\big|\psi_-(\|z-\mu-tv\|) - \psi_-(\|z-\mu\|)\big| \;&\leq\; \psi'_-(c)\big|\|z-\mu-tv\| - \|z-\mu\|\big| \\ &\leq\; \psi'_-(c) t\|v\|.\end{aligned}$$

Since convexity of ψ_- implies that $t \mapsto \psi'_-(t)$ is monotone non-decreasing on $(0, \infty)$, we have $\psi'_-(c) \leq \psi'_-(\max(\|z-\mu\|, \|z-\mu-tv\|)) \leq \psi'_-(\|z-\mu\|+r)$, which yields the desired inequality. □

Proof of Theorem 5.2. We first prove the result under Assumption (A). Let $\mathcal{N}$ be an open neighborhood of $\{\mu_0\}$ such that $P[\{\mu\}] + P[\|Z-\mu\| \in (0,\infty) \setminus \mathcal{D}_\rho] = 0$ for any $\mu \in \mathcal{N}$. Theorem 5.1 ensures that $M^\rho_{\alpha,u}$ is continuously differentiable on $\mathcal{N}$, with corresponding gradient (below, we may define $\nabla H^\rho_{\alpha,u}(z-\mu)$ arbitrarily at the z values where the gradient is undefined, since the collection of such z values has P-probability zero by assumption)

$$
\begin{aligned}
\nabla M^\rho_{\alpha,u}(\mu) &= -\int_{\mathbb{R}^d \setminus \{\mu\}} \nabla H^\rho_{\alpha,u}(z-\mu)\, dP(z) \\
&= \int_{\mathbb{R}^d} \bigg\{ -\psi_-(\|z-\mu\|)\bigg(1+\alpha \frac{u'(z-\mu)}{\|z-\mu\|}\bigg)\frac{z-\mu}{\|z-\mu\|}\xi_{z,\mu} \\
&\qquad -\alpha u \frac{\rho(\|z-\mu\|)}{\|z-\mu\|}\xi_{z,\mu} + \alpha \frac{\rho(\|z-\mu\|)}{\|z-\mu\|}\frac{u'(z-\mu)}{\|z-\mu\|}\frac{z-\mu}{\|z-\mu\|}\xi_{z,\mu}\bigg\}\, dP(z)
\end{aligned}
$$

for any $\mu \in \mathcal{N}$: see (S.5.27) and Lemma S.5.1(ii). Fix $v \in \mathbb{R}^d \setminus \{0\}$ and $t \in (0, t_0)$, where t_0 is such that $B(\mu_0, t_0\|v\|) \subset \mathcal{N}$. Let then

$$
\begin{aligned}
&f_1(z) = \psi_-(\|z-\mu_0\|),\ f_2(z) = 1 + \alpha u' \frac{z-\mu_0}{\|z-\mu_0\|},\ f_3(z) = \frac{z-\mu_0}{\|z-\mu_0\|}, \\
&g_1(z) = \frac{\rho(\|z-\mu_0\|)}{\|z-\mu_0\|},\ g_2(z) = \frac{u'(z-\mu_0)}{\|z-\mu_0\|},\ g_3(z) = \frac{z-\mu_0}{\|z-\mu_0\|}.
\end{aligned}
$$

Since P is non-atomic in $\mathcal{N}$, we have

$$
\begin{aligned}
&\frac{\nabla M^\rho_{\alpha,u}(\mu_0+tv) - \nabla M^\rho_{\alpha,u}(\mu_0)}{t} \\
&\quad = -\int_{\mathbb{R}^d \setminus \{\mu_0\}} \frac{\nabla H^\rho_{\alpha,u}(z-\mu_0-tv) - \nabla H^\rho_{\alpha,u}(z-\mu_0)}{t} \xi_{z,\mu_0+tv}\, dP(z) \\
&\quad = \int_{\mathbb{R}^d \setminus \{\mu_0\}} \big\{I_{t,1}(z) + I_{t,2}(z) + I_{t,3}(z)\big\}\xi_{z,\mu_0+tv}\, dP(z),
\end{aligned}
$$

where we let

$$
\begin{aligned}
I_{t,1}(z) &= \frac{f_1(z)f_2(z)f_3(z) - f_1(z-tv)f_2(z-tv)f_3(z-tv)}{t} \\
I_{t,2}(z) &= \alpha u\, \frac{g_1(z) - g_1(z-tv)}{t},
\end{aligned}
$$

and

$$
I_{t,3}(z) = \alpha\, \frac{g_1(z-tv)g_2(z-tv)g_3(z-tv) - g_1(z)g_2(z)g_3(z)}{t}.
$$

Decompose $I_{t,1}(z)$ into

$$I_{t,1}(z) = \frac{f_1(z) - f_1(z-tv)}{t} f_2(z-tv) f_3(z-tv) + f_1(z) \frac{f_2(z) - f_2(z-tv)}{t} f_3(z-tv) + f_1(z) f_2(z) \frac{f_3(z) - f_3(z-tv)}{t}.$$

Applying Lemma S.5.3(ii) and Lemma S.1.2, we obtain

$$\|I_{t,1}(z)\| \leq 2\|v\| \frac{\psi_-(\|z-\mu_0\|)}{\|z-\mu_0\|} + 2\|v\| \frac{\psi_-(\|z-\mu_0\|)}{\|z-\mu_0\|} + 4\|v\| \frac{\psi_-(\|z-\mu_0\|)}{\|z-\mu_0\|}$$

for any $z \in \mathbb{R}^d \setminus \{\mu_0, \mu_0 + tv\}$. Lemma S.5.3(i) directly yields

$$\|I_{t,2}(z)\| \leq \|v\| \frac{\psi_-(\|z-\mu_0\|)}{\|z-\mu_0\|}$$

for any $z \in \mathbb{R}^d \setminus \{\mu_0, \mu_0 + tv\}$. Finally, considering a decomposition similar as the one used for $I_{t,1}(z)$, Lemma S.5.3(i) and Lemma S.1.2 provide

$$\|I_{t,3}(z)\| \leq \|v\| \frac{\psi_-(\|z-\mu_0\|)}{\|z-\mu_0\|} + 2\|v\| \frac{\rho(\|z-\mu_0\|)}{\|z-\mu_0\|^2} + 2\|v\| \frac{\rho(\|z-\mu_0\|)}{\|z-\mu_0\|^2}$$

for any $z \in \mathbb{R}^d \setminus \{\mu_0, \mu_0 + tv\}$. Since $\rho(t)/t \leq \psi_-(t)$ for any $t > 0$, we conclude that $z \mapsto \|I_{t,1}(z) + I_{t,2}(z) + I_{t,3}(z)\|$ is upper-bounded P-almost everywhere by $z \mapsto C\|v\|\psi_-(\|z-\mu_0\|)/\|z-\mu_0\|$ (here, C is a positive real constant), which is a P-integrable function by assumption. For any $i = 1, \ldots, d$, Lebesgue's DCT therefore provides

$$\begin{aligned}
\lim_{t \overset{>}{\to} 0} & \frac{\partial_i M^{\rho}_{\alpha,u}(\mu_0 + tv) - \partial_i M^{\rho}_{\alpha,u}(\mu_0)}{t} \\
&= - \int_{\mathbb{R}^d \setminus \{\mu_0\}} \lim_{t \overset{>}{\to} 0} \frac{\partial_i H^{\rho}_{\alpha,u}(z - \mu_0 - tv) - \partial_i H^{\rho}_{\alpha,u}(z-\mu_0)}{t} \xi_{z,\mu_0+tv} \, dP(z) \\
&= \sum_{j=1}^d v_j \int_{\mathbb{R}^d \setminus \{\mu_0\}} \partial_j \partial_i H^{\rho}_{\alpha,u}(z-\mu_0) \, dP(z) = (\nabla^2 M^{\rho}_{\alpha,u}(\mu_0) v)_i, \qquad \text{(S.5.29)}
\end{aligned}$$

where the limit of the integrand exists for P-almost any $z \in \mathbb{R}^d$ (for the other values of z, the integrand in the last integral may be defined arbitrarily). The form of the Hessian matrix provided in the statement of the theorem then follows from Lemma S.5.1(ii).

It remains to show the result also holds when Assumption (A′) holds. The proof proceeds along the same lines, but the upper-bounding of $z \mapsto \|I_{t,1}(z) + I_{t,2}(z) + I_{t,3}(z)\|$

is now based on Lemma S.5.3(iii)–(iv). More precisely, Lemma S.5.3(iv) and Lemma S.1.2 entail that

$$\|I_{t,1}(z)\| \le 2\|v\|\psi'_-(\|z-\mu_0\|+r) + 2\|v\|\frac{\psi_-(\|z-\mu_0\|)}{\|z-\mu_0\|} + 4\|v\|\frac{\psi_-(\|z-\mu_0\|)}{\|z-\mu_0\|}$$

for any $z \in \mathbb{R}^d \setminus \{\mu_0, \mu_0+tv\}$ and $t \in (0, r/\|v\|]$. Lemma S.5.3(iii) directly yields

$$\|I_{t,2}(z)\| \le \|v\|\frac{\psi_-(\|z-\mu_0\|+r)}{\|z-\mu_0\|+r}$$

for any $z \in \mathbb{R}^d \setminus \{\mu_0, \mu_0+tv\}$ and $t \in (0, r/\|v\|]$. Finally, considering a decomposition similar as the one used for $I_{t,1}(z)$, Lemma S.5.3(iii) and Lemma S.1.2 provide

$$\|I_{t,3}(z)\| \le \|v\|\frac{\psi_-(\|z-\mu_0\|+r)}{\|z-\mu_0\|+r} + 2\|v\|\frac{\rho(\|z-\mu_0\|)}{\|z-\mu_0\|^2} + 2\|v\|\frac{\rho(\|z-\mu_0\|)}{\|z-\mu_0\|^2}$$

for any $z \in \mathbb{R}^d \setminus \{\mu_0, \mu_0+tv\}$ and $t \in (0, r/\|v\|]$. Since $\rho(t)/t^2 \le \psi_-(t)/t \le \psi'_-(t)$ for any $t > 0$ and $t \mapsto \psi'_-(t)$ is monotone non-decreasing on $(0,\infty)$, we conclude that $z \mapsto \|I_{t,1}(z) + I_{t,2}(z) + I_{t,3}(z)\|$ is upper-bounded P-almost everywhere by $z \mapsto C\|v\|\psi'_-(\|z-\mu_0\|+r)$ (here, C is a positive real constant), which is a P-integrable function by assumption. Consequently, we can still apply Lebesgue's DCT, which shows that (S.5.29) holds under Assumption (A′), too. □

S.6. Proofs for Section 6

Proof of Proposition 6.1. Let $(\alpha_n u_n)$ be a sequence in $\mathcal{B}^d_{\alpha_\rho}$ converging to $\alpha u \in \mathcal{B}^d_{\alpha_\rho}$. We want to show that $Q(\alpha_n u_n) \to Q(\alpha u)$, that is, $\mu^\rho_{\alpha_n,u_n} \to \mu^\rho_{\alpha,u}$. Since $M^\rho_{\alpha_n,u_n}(\mu^\rho_{\alpha,u}) \to M^\rho_{\alpha,u}(\mu^\rho_{\alpha,u})$ (Lemma S.2.1), $(M^\rho_{\alpha_n,u_n}(\mu^\rho_{\alpha,u}))$ is a bounded sequence. Since $M^\rho_{\alpha_n,u_n}(\mu^\rho_{\alpha_n,u_n}) \le M^\rho_{\alpha_n,u_n}(\mu^\rho_{\alpha,u})$ for any n by definition of ρ-quantiles, the sequence $(M^\rho_{\alpha_n,u_n}(\mu^\rho_{\alpha_n,u_n}))$ is upper-bounded. Since (α_n) is a sequence in $[0,\alpha_\rho)$, we have that

$$\limsup_{n\to\infty} \alpha_n u'_n \mu^\rho_{\alpha_n,u_n}/\|\mu^\rho_{\alpha_n,u_n}\| < 1\cdot$$

Lemma S.2.2 and the fact that $(M^\rho_{\alpha_n,u_n}(\mu^\rho_{\alpha_n,u_n}))$ is upper-bounded then entail that the sequence $(\mu^\rho_{\alpha_n,u_n})$ is bounded. Now, assume ad absurdum that $(\mu^\rho_{\alpha_n,u_n})$ does not converge to $\mu^\rho_{\alpha,u}$. Upon extraction of a subsequence, we may assume that there exists $\varepsilon > 0$ such that $\|\mu^\rho_{\alpha_n,u_n} - \mu^\rho_{\alpha,u}\| \ge \varepsilon$ for any n. This implies that no subsequence of $(\mu^\rho_{\alpha_n,u_n})$ converges to $\mu^\rho_{\alpha,u}$. However, since $(\mu^\rho_{\alpha_n,u_n})$ is a bounded sequence, it has a subsequence $(\mu^\rho_{\alpha_{n_k},u_{n_k}})$ that converges in $\mathbb{R}^d$, to v say. By taking limits as $k \to \infty$ in both sides of

$$M^\rho_{\alpha_{n_k},u_{n_k}}(\mu^\rho_{\alpha_{n_k},u_{n_k}}) \le M^\rho_{\alpha_{n_k},u_{n_k}}(\mu^\rho_{\alpha,u}),$$

Lemma S.2.1 then yields $M^{\rho}_{\alpha,u}(v) \leq M^{\rho}_{\alpha,u}(\mu_{\alpha,u})$. Since $\mu^{\rho}_{\alpha,u}$ is the unique minimizer of $M^{\rho}_{\alpha,u}$ (Theorem 3.3), we have $v = \mu^{\rho}_{\alpha,u}$, so that we identified a subsequence of $(\mu^{\rho}_{\alpha_n,u_n})$ that converges to $\mu^{\rho}_{\alpha,u}$. Since this is a contradiction, we conclude that $(\mu^{\rho}_{\alpha_n,u_n})$ converges to $\mu^{\rho}_{\alpha,u}$. □

Lemma S.6.1. *Let $\rho \in \mathcal{C}$ and assume that $P \in \mathcal{P}^d_\rho$ is not a Dirac probability measure. Then, for any $\mu \in \mathbb{R}^d$, the matrix $T(\mu)$ defined in Theorem 5.1 is positive definite, hence invertible.*

Proof of Lemma S.6.1. First note that, since $\rho(t)/t \leq \psi_-(t)$ for any $t > 0$, we have

$$\begin{aligned} v'T(\mu)v &= \mathrm{E}\bigg[\bigg\{\frac{\rho(\|Z-\mu\|)}{\|Z-\mu\|} \\ &\qquad\qquad + \bigg(\psi_-(\|Z-\mu\|) - \frac{\rho(\|Z-\mu\|)}{\|Z-\mu\|}\bigg)\frac{\{v'(Z-\mu)\}^2}{\|Z-\mu\|^2}\bigg\}\xi_{Z,\mu}\bigg] \\ &\geq \mathrm{E}\bigg[\frac{\rho(\|Z-\mu\|)}{\|Z-\mu\|}\xi_{Z,\mu}\bigg] \geq 0 \end{aligned}$$

for any $\mu \in \mathbb{R}^d$ and $v \in \mathcal{S}^{d-1}$. Now, assume ad absurdum that there exist $\mu \in \mathbb{R}^d$ and $v \in \mathcal{S}^{d-1}$ such that $v'T(\mu)v = 0$. We must then have

$$\int_{\mathbb{R}^d \setminus \{\mu\}} \frac{\rho(\|z-\mu\|)}{\|z-\mu\|}\, dP(z) = 0,$$

which, since $\rho(t) > 0$ for any $t > 0$, implies that $P[\{\mu\}] = 1$. This contradicts the assumption that P is not a Dirac probability measure. □

Proof of Theorem 6.1. Under the assumptions of the theorem, $\mu = Q(\alpha u)$, with $\alpha \in [0, \alpha_\rho)$ and $u \in \mathcal{S}^{d-1}$, implies that $R(\mu) = \alpha u$; see the discussion above the statement of the theorem. It directly follows that $R \circ Q$ is the identity map on $\mathcal{B}^d_{\alpha_\rho}$, which in turn entails that $Q : \mathcal{B}^d_{\alpha_\rho} \to \mathcal{Z}_\rho$ and $R : \mathcal{Z}_\rho \to \mathcal{B}^d_{\alpha_\rho}$ are one-to-one maps. Now, since $R(\mu) = (T(u))^{-1}v(\mu)$ for any $\mu \in \mathbb{R}^d$, continuity of R is a direct consequence of the fact that, under the assumptions considered, the maps $\mu \mapsto T(\mu)$ and $\mu \mapsto v(\mu)$ are continuous on $\mathbb{R}^d$; see the proof of Theorem 5.1. Since continuity of Q was already established in Proposition 6.1, the result is proved. □

The proof of Theorem 6.2 requires Lemma S.6.3 below, which itself relies on the following preliminary result.

Lemma S.6.2. *Let $\rho \in \mathcal{C}$ be such that $t \mapsto t^2/\rho(t)$ is concave on $(0,\infty)$. Then, $t \mapsto \rho(t)/t^2$ is convex and monotone non-increasing on $(0,\infty)$.*

Proof of Lemma S.6.2. Since $t \mapsto g(t) = t^2/\rho(t)$ is concave on $(0,\infty)$, it is left-differentiable on $(0,\infty)$ and the corresponding left-derivative, g'_- say, is monotone non-increasing. Therefore, for any $t_0 > 0$, Lemma S.1.3(i) yields

$$g(t) \le g(t_0) + g'_-(t_0)(t - t_0)$$

for any $t > t_0$. If $g'_-(t_0) < 0$ for some $t_0 > 0$, then it follows that $g(t)$ is strictly negative for t large, which is a contradiction. Thus, $g'_-(t) \ge 0$ for any $t > 0$. Lemma S.1.3(i) then implies that g is monotone non-decreasing on $(0,\infty)$, hence that $1/g$ is monotone non-increasing on $(0,\infty)$. It remains to show that $1/g$ is convex on $(0,\infty)$. To that end, fix $0 < s < t$ and $\lambda \in (0,1)$. Using concavity of g, we obtain by Jensen's inequality (for the convex function $z \mapsto 1/z$ and for the measure attributing weight $1-\lambda$ and λ to $g(s)$ and $g(t)$, respectively)

$$\frac{1}{g((1-\lambda)s + \lambda t)} \le \frac{1}{(1-\lambda)g(s) + \lambda g(t)} \le (1-\lambda)\frac{1}{g(s)} + \lambda \frac{1}{g(t)},$$

which establishes the result. □

Lemma S.6.3. *Let $\rho \in \mathcal{C}$ be such that $t \mapsto t^2/\rho(t)$ is concave on $(0,\infty)$. Assume that P is not concentrated on a line. Assume further that $\psi_+(0)P[\{\mu\}] + P[\|Z-\mu\| \in (0,\infty) \setminus \mathcal{D}_\rho] = 0$ for any $\mu \in \mathbb{R}^d$. Then, there is no $u \in \mathcal{S}^{d-1}$ for which the equation $\nabla M^\rho_{1,u}(\mu) = 0$ has a solution.*

Proof of Lemma S.6.3. Ad absurdum, let $u \in \mathcal{S}^{d-1}$ and $\mu \in \mathbb{R}^d$ be such that $\nabla M^\rho_{1,u}(\mu) = 0$. Thus, we must have $u'\nabla M^\rho_{1,u}(\mu) = 0$, which, in view of Theorem 5.1, rewrites

$$\mathrm{E}\bigg[\bigg\{\frac{\rho(\|Z-\mu\|)}{\|Z-\mu\|} + \bigg(\psi_-(\|Z-\mu\|) - \frac{\rho(\|Z-\mu\|)}{\|Z-\mu\|}\bigg)\frac{\{u'(Z-\mu)\}^2}{\|Z-\mu\|^2}\bigg\}\xi_{Z,\mu}\bigg]$$
$$+\mathrm{E}\bigg[\psi_-(\|Z-\mu\|)\frac{u'(Z-\mu)}{\|Z-\mu\|}\xi_{Z,\mu}\bigg] = 0.$$

Straightforward computations allow us to rewrite this as

$$\mathrm{E}\Big[g_1(\|Z-\mu\|)\zeta^2_{Z,\mu} + g_2(\|Z-\mu\|)\zeta_{Z,\mu}\Big] = 0, \tag{S.6.30}$$

where we let

$$\zeta_{Z,\mu} := \bigg(1 + \frac{u'(Z-\mu)}{\|Z-\mu\|}\bigg)\xi_{Z,\mu},$$

$$g_1(t) := \psi_-(t) - \frac{\rho(t)}{t} \quad \text{and} \quad g_2(t) := \frac{2\rho(t)}{t} - \psi_-(t)$$

for any $t > 0$. Lemma S.1.4 entails that $g_1(t) \geq 0$ for any $t > 0$. Since Lemma S.6.2 implies that $t \mapsto \rho(t)/t^2$, hence also $t \mapsto \ln(\rho(t)/t^2)$, is non-increasing over $(0, \infty)$, we have (here, we consider left-differentiation)

$$0 \geq \left(\ln\left(\frac{\rho(t)}{t^2}\right)\right)' = \frac{\psi_-(t)}{\rho(t)} - \frac{2}{t} = \frac{1}{\rho(t)}\left(\psi_-(t) - \frac{2\rho(t)}{t}\right)$$

for any $t > 0$, so that we also have $g_2(t) \geq 0$ for any $t > 0$. Moreover, since $g_1(t) + g_2(t) = \rho(t)/t > 0$ for any $t > 0$, we have $\max(g_1(t), g_2(t)) > 0$ for any $t > 0$. Since the assumption that P is not concentrated on a line ensures that $A := \{z \in \mathbb{R}^d : \zeta_{z,\mu} = 0\}$ satisfies

$$P[A] = P[Z - \mu \in \{cu : c \leq 0\}] \leq P[Z \in \{\mu + cu : c \in \mathbb{R}\}] < 1,$$

we thus have

$$\begin{aligned} &\mathrm{E}\Big[g_1(\|Z-\mu\|)\zeta^2_{Z,\mu} + g_2(\|Z-\mu\|)\zeta_{Z,\mu}\Big] \\ &\qquad = \int_{\mathbb{R}^d \setminus A} \big\{g_1(\|z-\mu\|)\zeta^2_{z,\mu} + g_2(\|z-\mu\|)\zeta_{z,\mu}\big\}\, dP(z) > 0, \end{aligned}$$

since $g_1(\|z-\mu\|)\zeta^2_{z,\mu} + g_2(\|z-\mu\|)\zeta_{z,\mu} > 0$ for any $z \notin A$ (the discussion above implies that the nonnegative quantities $g_1(\|z-\mu\|)$ and $g_2(\|z-\mu\|)$ cannot be both zero at the same z). Since this contradicts (S.6.30), the result is proved. □

Proof of Theorem 6.2. We first show that $R(\mathbb{R}^d) \subseteq \mathcal{B}^d$. To do so, assume, ad absurdum, that there exists $\mu \in \mathbb{R}^d$ such that $\|R(\mu)\| \geq 1$. We consider two cases. (a) $\|R(\mu)\| = 1$, so that $R(\mu) = u$ for some $u \in \mathcal{S}^{d-1}$. Then, $\nabla M^\rho_{1,u}(\mu) = T(\mu)(R(\mu) - u) = 0$, which, in view of Lemma S.6.3, is a contradiction. (b) $\|R(\mu)\| > 1$. Fix then arbitrarily $\alpha_0 u_0 \in \mathcal{B}^d$, and observe that $\|R(\mu^\rho_{\alpha_0,u_0})\| = \|\alpha_0 u_0\| = \alpha_0 < 1$. Recalling that R is continuous (Theorem 6.1), the intermediate value theorem then guarantees that there exists $\lambda \in (0,1)$ such that $\|R((1-\lambda)\mu^\rho_{\alpha_0,u_0} + \lambda\mu)\| = 1$, which, proceeding as in (a), provides a contradiction. Therefore, $R(\mathbb{R}^d) \subseteq \mathcal{B}^d$.

Now, fix $\mu \in \mathbb{R}^d$. Since $R(\mathbb{R}^d) \subseteq \mathcal{B}^d$, there exist $\alpha \in [0,1)$ and $u \in \mathcal{S}^{d-1}$ such that $R(\mu) = \alpha u$, which implies that $\mu = \mu^\rho_{\alpha,u} = Q(\alpha u)$. Therefore, $\mathcal{Z}_\rho = Q(\mathcal{B}^d) = \mathbb{R}^d$, and the result follows from Theorem 6.1. □

S.7. Proofs for Section 7

Proof of Theorem 7.1. (i) Assume, ad absurdum, that $\|\mu^\rho_{\alpha_n,u_n}\|$ does not diverge to ∞ as $n \to \infty$. The sequence $(\mu^{\rho}_{\alpha_n,u_n})$ then admits a subsequence that is bounded, hence a further subsequence, $(\mu^{\rho}_{\alpha_{n_\ell},u_{n_\ell}})$, that converges in $\mathbb{R}^d$. Let us denote as μ_* the corresponding limit. Since R is continuous, taking limits as $\ell \to \infty$ in both sides of $\|R(\mu^{\rho}_{\alpha_{n_\ell},u_{n_\ell}})\| = \|\alpha_{n_\ell} u_{n_\ell}\| = \alpha_{n_\ell}$ then provides $\|R(\mu_*)\| = 1$. Thus, $R(\mu_*) = v$ for some $v \in \mathcal{S}^{d-1}$, which implies that $\nabla M^{\rho}_{1,v}(\mu_*) = T(\mu_*)(R(\mu_*) - v) = 0$. Since this contradicts Lemma S.6.3, the result is proved. (ii) We now assume that u_n converges to $u(\in \mathcal{S}^{d-1})$. Consider an arbitrary subsequence $(\omega_k = \mu^{\rho}_{\alpha_{n_k},u_{n_k}})$ of $(\mu^{\rho}_{\alpha_n,u_n})$ and fix $\mu_0 \in \mathbb{R}^d$ arbitrarily. The continuity of $(\alpha, u) \mapsto M^{\rho}_{\alpha,u}(\mu_0)$ (Lemma S.2.1) implies that the sequence $(M^{\rho}_{\alpha_{n_k},u_{n_k}}(\mu_0))$ is bounded. Since, by definition,

$$M^{\rho}_{\alpha_{n_k},u_{n_k}}(\omega_k) \leq M^{\rho}_{\alpha_{n_k},u_{n_k}}(\mu_0)$$

for any k, the sequence $(M^{\rho}_{\alpha_{n_k},u_{n_k}}(\omega_k))$ is then upper-bounded, too. Assume now that

$$\limsup_{k\to\infty} u'_{n_k}\omega_k/\|\omega_k\| < 1.$$

Since $\|\omega_k\| \to \infty$, Lemma S.2.2 then implies that $M_{\alpha_{n_k},u_{n_k}}(\omega_k) \to \infty$ as $k \to \infty$, which contradicts the fact that the sequence $(M^{\rho}_{\alpha_{n_k},u_{n_k}}(\omega_k))$ is bounded. Therefore, we must have

$$\limsup_{k\to\infty} u'_{n_k}\omega_k/\|\omega_k\| = 1.$$

Since u_{n_k} converges to u, this entails that $\limsup_{k\to\infty} u'\omega_k/\|\omega_k\| = 1$, so that (ω_k) admits a subsequence (ω_{k_ℓ}) for which $u'_{n_{k_\ell}}\omega_{k_\ell}/\|\omega_{k_\ell}\| \to 1$ as $\ell \to \infty$. We thus proved that any subsequence of $(\mu^{\rho}_{\alpha_n,u_n}/\|\mu^{\rho}_{\alpha_n,u_n}\|)$ admits a further subsequence converging to u. This implies that $\mu^{\rho}_{\alpha_n,u_n}/\|\mu^{\rho}_{\alpha_n,u_n}\| \to u$ as $n \to \infty$. □

The proof of Theorem 7.2 requires the following result.

Lemma S.7.1. *Let the assumptions of Theorem 7.2 hold. Then, for any sequences* $(\alpha_n) \in [0, 1)$, $(u_n) \in \mathcal{S}^{d-1}$, *and* $(\mu_n) \in \mathbb{R}^d$ *such that* $\|\mu_n\| \to \infty$, *we have that* $M^{\rho}_{\alpha_n,u_n}(\mu_n) \to \infty$ *as* $n \to \infty$.

Proof of Lemma S.7.1. We prove the result by showing that any subsequence of $(M^{\rho}_{\alpha_n,u_n}(\mu_n))$ has a further subsequence converging to ∞. To do so, fix a subsequence (μ_{n_k}) of (μ_n). If $\limsup_{k\to\infty} \alpha_{n_k} u'_{n_k}\mu_{n_k}/\|\mu_{n_k}\| < 1$, then Lemma S.2.2 yields $M^{\rho}_{\alpha_{n_k},u_{n_k}}(\mu_{n_k}) \to \infty$ as $k \to \infty$. We may thus assume that $\limsup_{k\to\infty} \alpha_{n_k} u'_{n_k}\mu_{n_k}/\|\mu_{n_k}\|$ $= 1$. Take then a subsequence (k_ℓ) such that $\alpha_{n_{k_\ell}} u'_{n_{k_\ell}}\mu_{n_{k_\ell}}/\|\mu_{n_{k_\ell}}\| \to 1$ as $\ell \to \infty$. By abuse of notation, we write α_ℓ, u_ℓ and μ_ℓ instead of $\alpha_{n_{k_\ell}}$, $u_{n_{k_\ell}}$ and $\mu_{n_{k_\ell}}$, respectively. By compactness of $[0, 1]$ and $\mathcal{S}^{d-1}$, we can assume, up to further extraction of a subsequence, that $\alpha_\ell \to 1$, $u_\ell \to u \in \mathcal{S}^{d-1}$ and $u'_\ell\mu_\ell/\|\mu_\ell\| \to 1$ as $\ell \to \infty$.

For any $z \in \mathbb{R}^d$ and $v \in \mathcal{S}^{d-1}$, denote as $D_v(z) = \{z + \lambda v : \lambda \in \mathbb{R}\}$ the line through z with direction v, and note that the distance between $y (\in \mathbb{R}^d)$ and $D_v(z)$ is given by

$$d_v(y,z) = \|(I_d - vv')(z-y)\|. \tag{S.7.31}$$

It will then play a key role in the proof that the P-probability of

$$E := \Big\{ z \in \mathbb{R}^d : \liminf_{\ell\to\infty} d_{u_\ell}(\mu_\ell, z) > 0 \Big\}$$

is positive. To address this point, take $z_0 \in \mathbb{R}^d \backslash E$ (if no such z_0 exists, then $E = \mathbb{R}^d$ has P-probability one). Up to extraction of yet another subsequence (which we do again without changing the notation), we may assume that $\lim_{\ell\to\infty} d_{u_\ell}(\mu_\ell, z_0) = 0$. For any $z \notin D_u(z_0)$, the inequality

$$d_{u_\ell}(\mu_\ell, z) \geq d_{u_\ell}(z, z_0) - d_{u_\ell}(\mu_\ell, z_0)$$

(which readily follows from using the triangle inequality in (S.7.31)) yields

$$\liminf_{\ell\to\infty} d_{u_\ell}(\mu_\ell, z) \geq \lim_{\ell\to\infty} d_{u_\ell}(z, z_0) - \lim_{\ell\to\infty} d_{u_\ell}(\mu_\ell, z_0) = d_u(z, z_0) > 0.$$

This shows that $\mathbb{R}^d / D_u(z_0) \subset E$, so that the assumption that P is not concentrated on a line yields $P[E] \geq P[\mathbb{R}^d / D_u(z_0)] > 0$.

To proceed with the proof, we need to consider two cases, according to the assumptions considered.

(i) We assume first that $\rho(t)/t^2 \to \infty$ as $t \to \infty$ and that Condition (b) holds. Let us then write

$$M^\rho_{\alpha_\ell, u_\ell}(\mu_\ell) = \mathcal{J}_1(\alpha_\ell, u_\ell, \mu_\ell) + \mathcal{J}_2(\alpha_\ell, u_\ell, \mu_\ell) + \mathcal{J}_3(\alpha_\ell, u_\ell, \mu_\ell)$$

with

$$\mathcal{J}_1(\alpha, u, \mu) = \int_{\mathbb{R}^d} \big(\rho(\|z-\mu\|) - \rho(\|z\|)\big)\bigg(1 + \alpha \frac{u'(z-\mu)}{\|z-\mu\|}\bigg) \xi_{z,\mu}\, dP(z),$$

$$\begin{aligned}\mathcal{J}_2(\alpha, u, \mu) &= \int_{\mathbb{R}^d \backslash \{0,\mu\}} \rho(\|z\|) \bigg\{ \bigg(1 + \alpha \frac{u'(z-\mu)}{\|z-\mu\|}\bigg) \xi_{z,\mu} - \bigg(1 + \alpha \frac{u'z}{\|z\|}\bigg) \xi_{z,0} \bigg\}\, dP(z) \\ &= \int_{\mathbb{R}^d \backslash \{0\}} \alpha \rho(\|z\|) \bigg(\frac{u'(z-\mu)}{\|z-\mu\|} - \frac{u'z}{\|z\|} \bigg) \xi_{z,\mu}\, dP(z),\end{aligned}$$

and

$$\begin{aligned}\mathcal{J}_3(\alpha, u, \mu) &= \int_{\{0,\mu\}} \rho(\|z\|) \bigg\{ \bigg(1 + \alpha \frac{u'(z-\mu)}{\|z-\mu\|}\bigg) \xi_{z,\mu} - \bigg(1 + \alpha \frac{u'z}{\|z\|}\bigg) \xi_{z,0} \bigg\}\, dP(z) \\ &= -\rho(\|\mu\|)\bigg(1 + \alpha \frac{u'\mu}{\|\mu\|}\bigg) P[\{\mu\}].\end{aligned}$$

Let us observe that for any $z \in \mathbb{R}^d$ such that $\|z-\mu_\ell\| > (u'_\ell(z-\mu_\ell))^2$,

$$\begin{aligned}
1+\alpha_\ell \frac{u'_\ell(z-\mu_\ell)}{\|z-\mu_\ell\|} &= \frac{1-\alpha_\ell^2\Big(\frac{u'_\ell(z-\mu_\ell)}{\|z-\mu_\ell\|}\Big)^2}{1-\alpha_\ell \frac{u'(z-\mu_\ell)}{\|z-\mu_\ell\|}} \\
&= \frac{1-\Big(\frac{u'_\ell(z-\mu_\ell)}{\|z-\mu_\ell\|}\Big)^2 + \Big(\frac{u'_\ell(z-\mu_\ell)}{\|z-\mu_\ell\|}\Big)^2 (1-\alpha_\ell^2)}{1-\alpha_\ell \frac{u'_\ell(z-\mu_\ell)}{\|z-\mu_\ell\|}} \\
&= \frac{d^2_{u_\ell}(\mu_\ell, z)}{\|z-\mu_\ell\|^2\Big(1-\alpha_\ell \frac{u'_\ell(z-\mu_\ell)}{\|z-\mu_\ell\|}\Big)} + \Big(\frac{u'_\ell(z-\mu_\ell)}{\|z-\mu_\ell\|}\Big)^2 \frac{1-\alpha_\ell^2}{1-\alpha_\ell \frac{u'_\ell(z-\mu_\ell)}{\|z-\mu_\ell\|}} \\
&\geq \frac{d^2_{u_\ell}(\mu_\ell, z)}{\|z-\mu_\ell\|^2\Big(1-\alpha_\ell \frac{u'_\ell(z-\mu_\ell)}{\|z-\mu_\ell\|}\Big)}.
\end{aligned}$$

For any $z \in E$, we then have

$$\begin{aligned}
&\liminf_{\ell\to\infty} \rho(\|z-\mu_\ell\|)\Big(1+\alpha_\ell \frac{u'_\ell(z-\mu_\ell)}{\|z-\mu_\ell\|}\Big) \\
&\qquad \geq \liminf_{\ell\to\infty} \frac{\rho(\|z-\mu_\ell\|) d^2_{u_\ell}(\mu_\ell, z)}{\|z-\mu_\ell\|^2\Big(1-\alpha_\ell \frac{u'_\ell(z-\mu_\ell)}{\|z-\mu_\ell\|}\Big)} \\
&\qquad \geq \liminf_{\ell\to\infty} \frac{\rho(\|z-\mu_\ell\|)}{\|z-\mu_\ell\|^2} \times \liminf_{\ell\to\infty} \frac{d^2_{u_\ell}(\mu_\ell, z)}{1-\alpha_\ell \frac{u'_\ell(z-\mu_\ell)}{\|z-\mu_\ell\|}} \\
&\qquad = \infty,
\end{aligned}$$

as $\rho(t)/t^2 \to \infty$ as $t \to \infty$. Since the function

$$z \mapsto \big(\rho(\|z-\mu\|) - \rho(\|z\|)\big)\Big(1+\alpha \frac{u'(z-\mu)}{\|z-\mu\|}\Big)\xi_{z,\mu}$$

is lower-bounded by the P-integrable function $z \mapsto -2\rho(\|z\|)$ that does not depend on μ, Fatou's Lemma entails that

$$\begin{aligned}
&\liminf_{\ell\to\infty} \mathcal{J}_1(\alpha_\ell, u_\ell, \mu_\ell) \\
&\quad \geq \int_{\mathbb{R}^d} \liminf_{\ell\to\infty} \big(\rho(\|z-\mu_\ell\|) - \rho(\|z\|)\big)\bigg(1+\alpha_\ell \frac{u_\ell'(z-\mu_\ell)}{\|z-\mu_\ell\|}\bigg)\xi_{z,\mu_\ell}\, dP(z) \\
&\quad = \int_{\mathbb{R}^d} \liminf_{\ell\to\infty} \rho(\|z-\mu_\ell\|)\bigg(1+\alpha_\ell \frac{u_\ell'(z-\mu_\ell)}{\|z-\mu_\ell\|}\bigg)\xi_{z,\mu_\ell}\, dP(z) \\
&\quad \geq \int_{E} \liminf_{\ell\to\infty} \rho(\|z-\mu_\ell\|)\bigg(1+\alpha_\ell \frac{u_\ell'(z-\mu_\ell)}{\|z-\mu_\ell\|}\bigg)\xi_{z,\mu_\ell}\, dP(z) \\
&\quad = \infty
\end{aligned}$$

(recall that $P[E] > 0$). Observe next that, for any ℓ,

$$\big|\mathcal{J}_2(\alpha_\ell, u_\ell, \mu_\ell)\big| \leq 2\int_{\mathbb{R}^d} \rho(\|z\|)\, dP(z) < \infty$$

and

$$\big|\mathcal{J}_3(\alpha_\ell, u_\ell, \mu_\ell)\big| \leq 2\int_{\mathbb{R}^d} \rho(\|z\|)\, dP(z) < \infty.$$

Therefore, $\liminf_{\ell\to\infty} M^\rho_{\alpha_\ell,u_\ell}(\mu_\ell) = \infty$, so that $M^\rho_{\alpha_\ell,u_\ell}(\mu_\ell) \to \infty$ as $\ell \to \infty$, which proves the result.

(ii) We assume now that $\rho(t)/t^3$ is bounded away from 0 as $t \to \infty$ (but we do not assume anymore that Condition (b) holds). Write $M^\rho_{\alpha_\ell,u_\ell}(\mu_\ell)/\|\mu_\ell\| = \mathcal{I}_1(\alpha_\ell, u_\ell, \mu_\ell) + \mathcal{I}_2(\alpha_\ell, u_\ell, \mu_\ell) + \mathcal{I}_3(\alpha_\ell, u_\ell, \mu_\ell)$ as in (S.2.3). For any $z \in E$, we have

$$\begin{aligned}
&\liminf_{\ell\to\infty} \frac{\rho(\|z-\mu_\ell\|)}{\|\mu_\ell\|}\bigg(1+\alpha_\ell \frac{u_\ell'(z-\mu_\ell)}{\|z-\mu_\ell\|}\bigg) \\
&\qquad \geq \liminf_{\ell\to\infty} \frac{\rho(\|z-\mu_\ell\|) d^2_{u_\ell}(\mu_\ell, z)}{\|\mu_\ell\|\|z-\mu_\ell\|^2\Big(1-\alpha_\ell \frac{u_\ell'(z-\mu_\ell)}{\|z-\mu_\ell\|}\Big)} \\
&\qquad \geq \liminf_{\ell\to\infty} \frac{\rho(\|z-\mu_\ell\|)}{\|\mu_\ell\|\|z-\mu_\ell\|^2} \times \liminf_{\ell\to\infty} \frac{d^2_{u_\ell}(\mu_\ell, z)}{1-\alpha_\ell \frac{u_\ell'(z-\mu_\ell)}{\|z-\mu_\ell\|}} \\
&\qquad > 0,
\end{aligned}$$

since it is now assumed that $\liminf_{t\to\infty} \rho(t)/t^3 > 0$. For the same reason as in the proof of Lemma S.2.2, we may apply Fatou's Lemma for $\mathcal{I}_1$, which yields

$$\begin{aligned}
&\liminf_{\ell\to\infty} \mathcal{I}_1(\alpha_\ell, u_\ell, \mu_\ell) \\
&\quad \geq \int_{\mathbb{R}^d} \liminf_{\ell\to\infty} \frac{\rho(\|z-\mu_\ell\|)-\rho(\|z\|)}{\|\mu_\ell\|}\left(1+\alpha_\ell \frac{u_\ell'(z-\mu_\ell)}{\|z-\mu_\ell\|}\right)\xi_{z,\mu_\ell}\, dP(z) \\
&\quad = \int_{\mathbb{R}^d} \liminf_{\ell\to\infty} \frac{\rho(\|z-\mu_\ell\|)}{\|\mu_\ell\|}\left(1+\alpha_\ell \frac{u_\ell'(z-\mu_\ell)}{\|z-\mu_\ell\|}\right)\xi_{z,\mu_\ell}\, dP(z) \\
&\quad \geq \int_{E} \liminf_{\ell\to\infty} \frac{\rho(\|z-\mu_\ell\|)}{\|\mu_\ell\|}\left(1+\alpha_\ell \frac{u_\ell'(z-\mu_\ell)}{\|z-\mu_\ell\|}\right)\xi_{z,\mu_\ell}\, dP(z) \\
&\quad > 0.
\end{aligned}$$

Since the same arguments as in the proof of Lemma S.2.2 show that

$$\lim_{\ell\to\infty} \mathcal{I}_2(\alpha_\ell, u_\ell, \mu_\ell) = 0 \quad \text{and} \quad \liminf_{\ell\to\infty} \mathcal{I}_3(\alpha_\ell, u_\ell, \mu_\ell) \geq 0,$$

we obtain that $\liminf_{\ell\to\infty} M^\rho_{\alpha_\ell,u_\ell}(\mu_\ell)/\|\mu_\ell\| > 0$. Therefore, $M^\rho_{\alpha_\ell,u_\ell}(\mu_\ell) \to \infty$ as $\ell \to \infty$, which establishes the result. □

Proof of Theorem 7.2. Let (α_n) be a sequence in $[0,1)$ and (u_n) be a sequence in $\mathcal{S}^{d-1}$. Ad absurdum, assume that there exists a sequence $(\mu^\rho_{\alpha_n,u_n})$ of ρ-quantiles that exits any bounded set. In particular, $(\mu^\rho_{\alpha_n,u_n})$ is unbounded. With $\mu_0 \in \mathbb{R}^d$ fixed arbitrarily, we have

$$M^\rho_{\alpha_n,u_n}(\mu^\rho_{\alpha_n,u_n}) \leq M^\rho_{\alpha_n,u_n}(\mu_0)$$

for any n. From continuity of $(\alpha, u) \mapsto M^\rho_{\alpha,u}(\mu_0)$ and compactness of $[0,1]$ and $\mathcal{S}^{d-1}$, there exists $M > 0$ such that $M^\rho_{\alpha_n,u_n}(\mu_0) \leq M$ for any n, so that

$$M^\rho_{\alpha_n,u_n}(\mu^\rho_{\alpha_n,u_n}) \leq M \tag{S.7.32}$$

for any n. Since the sequence $(\mu^\rho_{\alpha_n,u_n})$ is unbounded, it admits a subsequence $(\mu^\rho_{\alpha_{n_k},u_{n_k}})$ such that $\|\mu^\rho_{\alpha_{n_k},u_{n_k}}\| \to \infty$ as $k \to \infty$. Lemma S.7.1 then implies that $M^\rho_{\alpha_n,u_n}(\mu^\rho_{\alpha_n,u_n}) \to \infty$ as $k \to \infty$, which contradicts (S.7.32). □

We now prove Proposition 7.1.

Proof of Proposition 7.1. Fix $u \in \mathcal{S}^{d-1}$ and let (α_n) be a sequence in $[0,1)$ that converges to one. Let $\mu^\rho_{\alpha_n,u}$ be a sequence of ρ-quantiles and let $\mu_0 \in \mathbb{R}^d$ be fixed. We have

$$M^\rho_{\alpha_n,u}(\mu^\rho_{\alpha_n,u}) \leq M^\rho_{\alpha_n,u}(\mu_0)$$

for any n. Since $(\mu^\rho_{\alpha_n,u})$ is bounded (Theorem 7.2), it admits a subsequence $(\mu^\rho_{\alpha_{n_k},u})$ that converges in $\mathbb{R}^d$, to v say. Continuity of $(\alpha, u, \mu) \mapsto M^\rho_{\alpha,u}(\mu)$ on $[0,1] \times \mathcal{S}^{d-1} \times \mathbb{R}^d$ implies that

$$M^\rho_{1,u}(v) \leq M^\rho_{1,u}(\mu_0)$$

for any n. Since this holds for any $\mu_0 \in \mathbb{R}^d$, we conclude that v minimizes $\mu \mapsto M^\rho_{1,u}(\mu)$ over $\mathbb{R}^d$, which establishes the result. □

Proof of Corollary 7.1. Let (α_n) be a sequence in $[0,1)$ that converges to $\alpha \in [0,1]$ and (u_n) be a sequence in $\mathcal{S}^{d-1}$ that converges to u. Let $(\mu^\rho_{\alpha_n,u_n})$ be a sequence of ρ-quantiles and $(\mu^\rho_{\alpha_{n_k},u_{n_k}})$ be a convergent subsequence. Denote as v its limit. For an arbitrarily fixed $\mu_0 \in \mathbb{R}^d$, we then have

$$M^\rho_{\alpha_{n_k},u_{n_k}}(\mu^\rho_{\alpha_{n_k},u_{n_k}}) \leq M^\rho_{\alpha_{n_k},u_{n_k}}(\mu_0)$$

for any k. Continuity of $(\alpha, u, \mu) \mapsto M^\rho_{\alpha,u}(\mu)$ on $[0,1] \times \mathcal{S}^{d-1} \times \mathbb{R}^d$ implies that

$$M^\rho_{\alpha,u}(v) \leq M^\rho_{\alpha,u}(\mu_0)$$

for any n. Since μ_0 is arbitrary, v minimizes $\mu \mapsto M^\rho_{\alpha,u}$, hence is a ρ-quantile of order α in direction u. (ii) Assume that $\mu \mapsto M^\rho_{\alpha,u}$ has a unique minimizer $\mu^\rho_{\alpha,u}$ and let now $(\mu^\rho_{\alpha_{n_k},u_{n_k}})$ be an arbitrary subsequence of $(\mu^\rho_{\alpha_n,u_n})$. This subsequence is bounded (Theorem 7.2), hence admits a converging subsequence. Part (i) of the result implies that the corresponding limit must be the unique minimizer $\mu^\rho_{\alpha,u}$ of $\mu \mapsto M^\rho_{\alpha,u}(\mu)$. We thus proved than any subsequence of $(\mu^\rho_{\alpha_n,u_n})$ admits a further subsequence converging to $\mu^\rho_{\alpha,u}$. This entails that $\mu^\rho_{\alpha_n,u_n} \to \mu^\rho_{\alpha,u}$ as $n \to \infty$. □

S.8. Proofs for Section 8

Proof of Theorem 8.1. The result follows by applying Theorem 1 in Niemiro (1992), as Theorem 2.1 and Theorem 3.3 show that the conditions (i)–(iii) in page 1515 of that paper are satisfied. □

We will need Lemmas S.8.2–S.8.4 below to prove Theorem 8.2. The first of these lemmas in turn requires the following preliminary result.

Lemma S.8.1. *Let $\rho \in \mathcal{C}$. Fix $\alpha \in [0, \alpha_\rho) \cup \{0\}$ and $u \in \mathcal{S}^{d-1}$. Fix $v \in \mathcal{S}^{d-1}$ and $x \in \mathbb{R}^d$ with $\|x\| \in \mathcal{D}_\rho$ such that $v'\nabla^2 H^\rho_{\alpha,u}(x)v = 0$. Then, $|v'x| = \|x\|$ and $\psi'_-(\|x\|) = 0$.*

Proof of Lemma S.8.1. As seen in the proof of Theorem 3.2,

$$\frac{t^2}{\rho(t)} v' \nabla^2 H^{\rho}_{\alpha,u}(x) v = g_{t,\alpha}(\theta,\omega) = i_t(\omega) + \alpha s_t(\theta,\omega),$$

where $t = \|x\|$, $\cos\theta = u'x/\|x\|$, $\cos\omega = v'x/\|x\|$, and

$$\begin{aligned} i_t(\omega) &= \frac{t\psi_-(t)}{\rho(t)} + \frac{t^2\psi'_-(t) - t\psi_-(t)}{\rho(t)}(\cos\omega)^2 \\ &= \frac{t^2}{\rho(t)}\left(\frac{\psi_-(t)}{t}(1-(\cos\omega)^2) + \psi'_-(t)(\cos\omega)^2\right) \geq 0 \end{aligned}$$

(we will not need the expression of $s_t(\theta,\omega)$ here). We consider two cases. (a) $\alpha = 0$. Since $v'\nabla^2 H^{\rho}_{\alpha,u}(x)v = 0$, we have $i_t(\omega) = 0$, which yields

$$\frac{\psi_-(t)}{t}(1-(\cos\omega)^2) = 0 \quad \text{and} \quad \psi'_-(t)(\cos\omega)^2 = 0.$$

Since $\psi_-(t) \geq \rho(t)/t > 0$ for any $t > 0$ (Lemma S.1.4), we must then have $(\cos\omega)^2 = 1$ and $\psi'_-(t) = 0$, which shows that the result holds in this case. (b) $\alpha > 0$ (so that $0 < \alpha < \alpha_\rho$). Since $v'\nabla^2 H^{\rho}_{\alpha,u}(x)v = 0$, we have $g_{t,\alpha}(\theta,\omega) = 0$. Ad absurdum, assume then that $(\cos\omega)^2 \neq 1$ or $\psi'_-(t) > 0$. Then, $i_t(\omega) > 0$, so that we must have $s_t(\theta,\omega) < 0$. Therefore, any $\alpha_0 > \alpha$ will provide $g_{t,\alpha_0}(\theta,\omega) < 0$, which implies $\alpha_0 > \alpha_\rho$. We thus proved that any $\alpha_0 > \alpha$ satisfies $\alpha_0 > \alpha_\rho$, which contradicts the assumption that $\alpha < \alpha_\rho$. Therefore, $(\cos\omega)^2 = 1$ and $\psi'_-(t) = 0$ in case (b), too. □

Lemma S.8.2. *Let the assumptions of Theorem 5.2 hold and assume further that* $P[E_{\mu_0,v}] < 1$ *for any* $v \in \mathcal{S}^{d-1}$, *with*

$$E_{\mu_0,v} := \{z \in \mathbb{R}^d : \psi'_-(\|z-\mu_0\|) = 0 \text{ and } z \in L_{\mu_0}(v)\},$$

where $L_{\mu_0}(v) = \{\mu_0 + \lambda v : \lambda \in \mathbb{R}\}$ *is the line through* μ_0 *with direction* v. *Then, for any* $\alpha \in [0,\alpha_\rho) \cup \{0\}$ *and* $u \in \mathcal{S}^{d-1}$, *the Hessian matrix* $\nabla^2 M^{\rho}_{\alpha,u}(\mu_0)$ *is positive definite.*

Proof of Lemma S.8.2. Fix $\alpha \in [0,\alpha_\rho) \cup \{0\}$ and $u \in \mathcal{S}^{d-1}$. Under the assumptions considered, it follows from (S.5.29) that

$$\nabla^2 M^{\rho}_{\alpha,u}(\mu_0) = \int_G \nabla^2 H^{\rho}_{\alpha,u}(z-\mu_0)\, dP(z), \qquad \text{(S.8.33)}$$

where $G := \{z \in \mathbb{R}^d : \|z-\mu_0\| \notin \mathcal{D}_\rho\}$ has probability one (so that $\nabla^2 H^{\rho}_{\alpha,u}(z-\mu_0)$ may be defined arbitrarily for $z \notin G$). Assume then, ad absurdum, that $v'\nabla^2 M^{\rho}_{\alpha,u}(\mu_0)v = 0$ for some $v \in \mathcal{S}^{d-1}$. It directly follows from (S.8.33) and convexity of $H^{\rho}_{\alpha,u}$ that $v'\nabla^2 H^{\rho}_{\alpha,u}(z-\mu_0)v = 0$ for any $z \in G$. Lemma S.8.1 thus implies that $\psi'_-(\|z-\mu_0\|) = 0$ and $z \in L_{\mu_0}$ for P-almost all $z \in \mathbb{R}^d$, a contradiction. □

Lemma S.8.3. *Let $\rho \in \mathcal{C}$ and $P \in \mathcal{P}^{\rho}_d$. Fix $\alpha \in [0,1)$, $u \in \mathcal{S}^{d-1}$, and $\mu \in \mathbb{R}^d$. Assume that*

$$\int_{\mathbb{R}^d} \psi^2_-(\|z-\mu\|)\, dP(z) < \infty.$$

Let Z be a random d-vector with distribution P. Then, the $d \times d$ matrix

$$\mathrm{E}[(\nabla H^{\rho}_{\alpha,u}(Z-\mu))(\nabla H^{\rho}_{\alpha,u}(Z-\mu))'\mathbb{I}[\|Z-\mu\| \in \mathcal{D}_\rho]]$$

exists and is finite.

Proof of Lemma S.8.3. Fix $r \in \{1,\ldots,d\}$. Lemma S.5.1(ii) readily entails that, if $\|z-\mu\| \in \mathcal{D}_\rho$, then

$$(\partial_r H^{\rho}_{\alpha,u}(z-\mu))^2 \leq C\bigg(\psi^2_-(\|z-\mu\|) + \frac{\rho^2(\|z-\mu\|)}{\|z-\mu\|^2}\bigg)$$

for some positive constant C. Since $\rho(t)/t \leq \psi_-(t)$ for any $t > 0$ (Lemma S.1.4), this implies that

$$\mathrm{E}[(\partial_r H^{\rho}_{\alpha,u}(Z-\mu))^2\mathbb{I}[\|Z-\mu\| \in \mathcal{D}_\rho]] < \infty.$$

Since this hols for any $r \in \{1,\ldots,d\}$, the result follows from the Cauchy–Schwarz inequality. □

Lemma S.8.4. *Let the assumptions of Theorem 8.2 hold and write $F = \{z \in \mathbb{R}^d : \|z-\mu^{\rho}_{\alpha,\mu}\| \in \mathcal{D}_\rho\}$. Then, for any $h \in \mathbb{R}^d$,*

$$R_n := \sum_{i=1}^n \big\{H^{\rho}_{\alpha,u}\big(Z_i - \mu^{\rho}_{\alpha,u} - \tfrac{h}{\sqrt{n}}\big) - H^{\rho}_{\alpha,u}(Z_i - \mu^{\rho}_{\alpha,u})\big\}\mathbb{I}[Z_i \in F]$$
$$+\frac{1}{\sqrt{n}}\sum_{i=1}^n h'\nabla H^{\rho}_{\alpha,u}(Z_i - \mu^{\rho}_{\alpha,u})\mathbb{I}[Z_i \in F]$$
$$-\frac{1}{2n}\sum_{i=1}^n h'\nabla^2 H^{\rho}_{\alpha,u}(Z_i - \mu^{\rho}_{\alpha,u})h\mathbb{I}[Z_i \in F]$$

converges to 0 in the $L_1(P)$ sense as $n \to \infty$.

Proof of Lemma S.8.4. For $z \in \mathbb{R}^d$, let

$$J_n(z) = n\big\{H^\rho_{\alpha,u}\big(z - \mu^\rho_{\alpha,u} - \tfrac{h}{\sqrt{n}}\big) - H^\rho_{\alpha,u}(z - \mu^\rho_{\alpha,u}) + \tfrac{h'}{\sqrt{n}}\nabla H^\rho_{\alpha,u}(z - \mu^\rho_{\alpha,u})\big\}\mathbb{I}[z \in F]$$

and $J(z) = \frac{1}{2}h'\nabla^2 H^\rho_{\alpha,u}(z - \mu^\rho_{\alpha,u})h\mathbb{I}[z \in F]$. First note that $J_n(z) \to J(z)$ as $n \to \infty$ for any $z \in \mathbb{R}^d$ (this is trivial for $z \notin F$ and results from Lemma S.5.1(ii) for $z \in F$). Since $P[F] = 1$ by assumption, observe that (S.5.27) and (S.5.29) yield

$$E[J_n(Z_1)] - E[J(Z_1)] = n\Big\{M^\rho_{\alpha,u}(\mu^\rho_{\alpha,u} + \tfrac{h}{\sqrt{n}}) - M^\rho_{\alpha,u}(\mu^\rho_{\alpha,u}) - \tfrac{1}{\sqrt{n}}h'\nabla M^\rho_{\alpha,u}(\mu^\rho_{\alpha,u}) - \tfrac{1}{2n}h'\nabla^2 M^\rho_{\alpha,u}(\mu^\rho_{\alpha,u})h\Big\}.$$

Since $M^\rho_{\alpha,u}$ is twice differentiable at $\mu^\rho_{\alpha,u}$ (Theorem 5.2), it follows that $E[J_n(Z_1)] \to E[J(Z_1)]$ as $n \to \infty$. Now, recalling that $\alpha \in [0,\alpha_\rho)\cup\{0\}$, the map $z \mapsto H^\rho_{\alpha,u}(z)$ is convex, which implies that $J_n(z)$ and $J(z)$ are nonnegative for any $z \in \mathbb{R}^d$. Therefore, Scheffé's lemma entails that $J_n(Z_1) \to J(Z_1)$ in the $L_1(P)$ sense as $n \to \infty$, so that $E[|R_n|] \leq E[|J_n(Z_1) - J(Z_1)|] = o(1)$ as $n \to \infty$. □

Proof of Theorem 8.2. Throughout the proof, $M^{\rho;P_n}_{\alpha,u}(\mu)$ stands for the objective function $M^\rho_{\alpha,u}$ associated to the empirical probability measure P_n, that is

$$M^{\rho;P_n}_{\alpha,u}(\mu) := \frac{1}{n}\sum_{i=1}^n \Big(H^\rho_{\alpha,u}(Z_i - \mu) - H^\rho_{\alpha,u}(Z_i)\Big),$$

for all $\mu \in \mathbb{R}^d$. We prove the result by applying Theorem 3 from Arcones (1998) with the sequence of stochastic processes $\{G_n(\theta) = nM^{\rho;P_n}_{\alpha,u}(\mu) : \theta = \mu \in \Theta = \mathbb{R}^d\}$, the fixed parameter value $\theta_0 = \mu^\rho_{\alpha,u}$, and the sequence of estimators $\hat\theta_n = \hat\mu^\rho_{\alpha,u}$. We now check that under the assumption of Theorem 8.2, Conditions (i)–(v) from Arcones' Theorem hold with

$$\eta_n := -\frac{1}{\sqrt{n}}\sum_{i=1}^n \nabla H^\rho_{\alpha,u}(Z_i - \mu^\rho_{\alpha,u})\mathbb{I}[Z_i \in F]$$

(as in Lemma S.8.4, $F = \{z \in \mathbb{R}^d : \|z - \mu^\rho_{\alpha,\mu}\| \in \mathcal{D}_\rho\}$), $V_n := \frac{1}{2}A = \frac{1}{2}\nabla^2 M^\rho_{\alpha,u}(\mu^\rho_{\alpha,u})$, and $M_n = \sqrt{n}I_d$. The restriction to $\alpha \in [0,\alpha_\rho)\cup\{0\}$ ensures that the convexity requirement in Condition (i) holds, whereas Condition (ii) directly follows from the fact that, by definition, $\hat\mu^\rho_{\alpha,u}$ minimizes $\mu \mapsto nM^{\rho;P_n}_{\alpha,u}(\mu)$. We have $P[F] = 1$ by assumption, so that (S.5.27) yields

$$\mathrm{E}[\nabla H^\rho_{\alpha,u}(Z_i - \mu^\rho_{\alpha,u})\mathbb{I}[Z_i \in F]] = -\nabla M^\rho_{\alpha,u}(\mu^\rho_{\alpha,u}) = 0$$

for any $i = 1,\ldots,n$. In view of Lemma S.8.3, the multivariate central limit theorem thus entails that, under the assumptions considered, η_n is asymptotically d-variate normal with mean vector zero and covariance matrix B. Consequently, Condition (iv) holds. As for Condition (v), it trivially follows from the fact that $V_n = \frac{1}{2}\nabla^2 M^{\rho}_{\alpha,u}(\mu^{\rho}_{\alpha,u})$ is positive definite (Lemma S.8.2) and does not depend on n.

Therefore, it only remains to show that Condition (iii) holds, that is, that, for each $h \in \mathbb{R}^d$,

$$nM^{\rho,P_n}_{\alpha,u}(\mu^{\rho}_{\alpha,u} + \tfrac{h}{\sqrt{n}}) - nM^{\rho,P_n}_{\alpha,u}(\mu^{\rho}_{\alpha,u}) - h'\eta_n - h'V_n h = o_{\mathrm{P}}(1) \tag{S.8.34}$$

as $n \to \infty$. In order to do so, write

$$\begin{aligned}
&nM^{\rho,P_n}_{\alpha,u}(\mu^{\rho}_{\alpha,u} + \tfrac{h}{\sqrt{n}}) - nM^{\rho,P_n}_{\alpha,u}(\mu^{\rho}_{\alpha,u})\\
&\quad= \sum_{i=1}^n \big\{H^{\rho}_{\alpha,u}\big(Z_i - \mu^{\rho}_{\alpha,u} - \tfrac{h}{\sqrt{n}}\big) - H^{\rho}_{\alpha,u}(Z_i - \mu^{\rho}_{\alpha,u})\big\}\mathbb{I}[Z_i \notin F]\\
&\qquad+ \sum_{i=1}^n \big\{H^{\rho}_{\alpha,u}\big(Z_i - \mu^{\rho}_{\alpha,u} - \tfrac{h}{\sqrt{n}}\big) - H^{\rho}_{\alpha,u}(Z_i - \mu^{\rho}_{\alpha,u})\big\}\mathbb{I}[Z_i \in F].
\end{aligned}$$

Note that

$$\sum_{i=1}^n \big\{H^{\rho}_{\alpha,u}\big(Z_i - \mu^{\rho}_{\alpha,u} - \tfrac{h}{\sqrt{n}}\big) - H^{\rho}_{\alpha,u}(Z_i - \mu^{\rho}_{\alpha,u})\big\}\mathbb{I}[Z_i \notin F] = o_{\mathrm{P}}(1)$$

as $n \to \infty$, since $P[F] = 1$ implies that, for any $\varepsilon > 0$,

$$\begin{aligned}
&P\bigg[\bigg|\sum_{i=1}^n \big\{H^{\rho}_{\alpha,u}\big(Z_i - \mu^{\rho}_{\alpha,u} - \tfrac{h}{\sqrt{n}}\big) - H^{\rho}_{\alpha,u}(Z_i - \mu^{\rho}_{\alpha,u})\big\}\mathbb{I}[Z_i \notin F]\bigg| > \varepsilon\bigg]\\
&\qquad\leq P\bigg[\bigcup_{i=1}^n [Z_i \notin F]\bigg] \leq \sum_{i=1}^n P[\mathbb{R}^d \setminus F] = 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&nM^{\rho,P_n}_{\alpha,u}(\mu^{\rho}_{\alpha,u} + \tfrac{h}{\sqrt{n}}) - nM^{\rho,P_n}_{\alpha,u}(\mu^{\rho}_{\alpha,u}) - h'\eta_n - h'V_n h\\
&\quad= \sum_{i=1}^n \big\{H^{\rho}_{\alpha,u}\big(Z_i - \mu^{\rho}_{\alpha,u} - \tfrac{h}{\sqrt{n}}\big) - H^{\rho}_{\alpha,u}(Z_i - \mu^{\rho}_{\alpha,u})\big\}\mathbb{I}[Z_i \in F]\\
&\qquad -h'\eta_n - \frac{1}{2n}\sum_{i=1}^n h'(\nabla^2 H^{\rho}_{\alpha,u}(Z_i - \mu^{\rho}_{\alpha,u})h\mathbb{I}[Z_i \in F]\\
&\qquad -\frac{1}{2}h'\bigg(A - \frac{1}{n}\sum_{i=1}^n \nabla^2 H^{\rho}_{\alpha,u}(Z_i - \mu^{\rho}_{\alpha,u})\mathbb{I}[Z_i \in F]\bigg)h + o_{\mathrm{P}}(1).
\end{aligned}$$

Applying Lemma S.8.4 and the law of large numbers thus establishes (S.8.34), which shows that Arcones' Condition (iii) holds, too.

Theorem 3 from Arcones (1998) therefore applies and yields that, as $n \to \infty$,

$$\sqrt{n}(\hat{\mu}^{\rho}_{\alpha,u} - \mu^{\rho}_{\alpha,u}) = -\frac{1}{2} V_n^{-1} \eta_n + o_{\mathrm{P}}(1) = -A^{-1} \eta_n + o_{\mathrm{P}}(1),$$

which is the desired result. The asymptotic normal distribution of $\sqrt{n}(\hat{\mu}^{\rho}_{\alpha,u} - \mu^{\rho}_{\alpha,u})$ then readily follows from the fact that, as already mentioned, η_n is asymptotically normal with mean vector zero and covariance matrix B. □

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