EXPLICIT RECOVERY OF A PROBABILITY MEASURE FROM ITS GEOMETRIC DEPTH

DIMITRI KONEN

Abstract. We prove that in any Euclidean space $\mathbb{R}^d$, an arbitrary probability measure P can be reconstructed explicitly by its geometric rank R_P. The reconstruction takes the form $P = \mathcal{L}_d(R_P)$, where $\mathcal{L}_d$ is a (potentially fractional) linear differential operator given in closed form. While the above equality holds in the sense of distributions for an arbitrary P, when P admits a density f_P we provide sufficient conditions to ensure that $f_P = \mathcal{L}_d(R_P)$ holds pointwise. Surprisingly, the reconstruction procedure is of a local nature when d is odd, and of a non-local nature when d is even. We give examples of the reconstruction in $\mathbb{R}^2$ and $\mathbb{R}^3$. We use our results to characterise the regularity of depth contours. We conclude the paper with a partial counterpart to the non-localisability in even dimensions.

Contents

1. Introduction

The cdf and quantile maps of univariate distributions play a vital role in statistics and probability. For instance, statistical procedures that combine broad validity (no need for moment assumption, resistance to possible outlying observations) and efficiency are typically based on ranks, computed by evaluating the cdf at observed data points. That is, each data point is used only through the fact it is the smallest one, or the second smallest one, etc. These procedures have found many applications in hypothesis testing, outliers detection, and extreme value theory, for instance. In the multivariate setting $\mathbb{R}^d$ with $d \geq 2$, however, no canonical ordering is available, so that there is no natural concept of ranks that can be used to define rank-based statistical procedures. In this context, *statistical depth* is a general device that allows one to define a center-outward ordering of data points in $\mathbb{R}^d$, thus providing the basis for the definition of suitable multivariate rank-based procedures. Since the introduction of the celebrated *halfspace depth* in [1], and the subsequent work of Regina Liu that made statistical depth a field in itself ([2], [3]), statisticians have proposed many concepts, called *multivariate quantiles, ranks and depth*, to extend these ideas to a multivariate framework. The most celebrated depths are the aforementioned halfspace depth, the *simplicial depth* [2], the *geometric (or spatial) depth* [4], and the *projection depth* [5]. Other approaches have been adopted, attempting to define a proper notion of multivariate cdf and quantiles; the most notable are based on regression quantiles [6], or optimal transport [7]. We also refer the reader to [8] for a review on the topic.

Among the concepts extending cdf's and quantiles to a multivariate setting, a popular approach is that of *geometric* multivariate ranks and quantiles, introduced in [9], on which this paper focuses. They enjoy important advantages over other competing approaches. Among them, let us stress that geometric ranks are available in closed form, which leads to trivial evaluation in the empirical case, unlike most competing concepts. As a consequence, explicit Bahadur-type representations and asymptotic normality results are provided in [9] and [10], when competing approaches offer at best consistency results only. Geometric ranks and quantiles also allow for direct extensions in infinite-dimensional Hilbert spaces ; see, e.g., [11] and [12]. We refer the reader to [13] for an overview of the scope of applications geometric ranks offer.

Let us briefly state standard definitions and results about geometric quantiles and ranks. We start by introducting a function strongly related to geometric quantiles. We denote the Euclidean inner product between two vectors $u = (u_1, \dots, u_d) \in \mathbb{R}^d$ and $v = (v_1, \dots, v_d) \in \mathbb{R}^d$ by $(u, v) := \sum_{i=1}^d u_i v_i$, and we let $|u| := \sqrt{(u, u)}$ stand for the Euclidean norm. We let $^{d-1} = \{x \in \mathbb{R}^d : (x, x) = 1\}$ be the unit sphere of $\mathbb{R}^d$.

Definition 1.1. *Let $d \geq 1$ and P a Borel probability measure on $\mathbb{R}^d$. We define the map $g_P : \mathbb{R}^d \to \mathbb{R}$ by letting*

$$\forall\, x \in \mathbb{R}^d, \qquad g_P(x) = \int_{\mathbb{R}^d} \Big(|z - x| - |z| \Big)\, dP(z).$$

The triangle inequality entails that g_P is well-defined, irrespective of the probability measure P, without any moment assumption. It is further easy to see that g_P is continuous over $\mathbb{R}^d$. Theorem 5.1 in [14] entails that g_P is continuously differentiable over an open subset $U \subset \mathbb{R}^d$ if and only if P has no atoms in U; in this case, we have

$$\forall\, x \in \mathbb{R}^d, \qquad \nabla g_P(x) = \int_{\mathbb{R}^d \setminus \{x\}} \frac{x - z}{|x - z|}\, dP(z).$$

Let us now turn to the definition of multivariate geometric quantiles, which are directional in nature : they are indexed by an order $\alpha \in [0, 1)$ and a direction $u \in {}^{d-1}$. In the univariate case $d = 1$, it is shown in [15] that geometric quantiles of order $\alpha \in [0, 1)$ in direction $u \in \{-1, +1\}$ reduce to the usual quantiles of order $(\alpha u + 1)/2 \in [0, 1)$.

Definition 1.2. *Let $d \geq 1$ and P a probability measure on $\mathbb{R}^d$. A geometric quantile of order $\alpha \in [0, 1)$ in direction $u \in {}^{d-1}$ for P is an arbitrary minimizer of the objective function*

$$x \mapsto O^P_{\alpha, u}(x) := g_P(x) - (\alpha u, x)$$

over $\mathbb{R}^d$.

As explainded before Definition 1.2, we have $\nabla O^P_{\alpha,u}(x) = \nabla g_P(x) - \alpha u$ for any $x \in \mathbb{R}^d$ provided P has no atoms. Further requiring that P is not supported on a single line of $\mathbb{R}^d$, it is proved in [16] that $O^P_{\alpha,u}$ is strictly convex over $\mathbb{R}^d$ and, therefore, that geometric quantiles of order α in direction u for P are unique for any $\alpha \in [0, 1)$ and $u \in {}^{d-1}$; we write such a quantile as $Q_P(\alpha u)$. In particular, $Q_P(\alpha u)$ is the unique solution $x \in \mathbb{R}^d$ to the equation

$$\nabla g_P(x) = \alpha u.$$

Under these assumptions, Theorem 6.2 in [14] entails that the quantile map $\alpha u \mapsto Q_P(\alpha u)$ is invertible with inverse $Q_P^{-1} = \nabla g_P$. In anology with the univariate case, the equality $Q_P^{-1} = \nabla g_P$ provides the motivation to define ∇g_P as a natural multivariate analog of the cdf, which we call *geometric rank*.

Definition 1.3. *Let $d \geq 1$ and P a Borel probability measure on $\mathbb{R}^d$. The geometric rank R_P of P is the map $R_P : \mathbb{R}^d \to \mathbb{R}^d$ defined by letting*

$$R_P(x) = \int_{\mathbb{R}^d \setminus \{x\}} \frac{x - z}{|x - z|} \, dP(z), \qquad \forall\, x \in \mathbb{R}^d.$$

Notice that R_P is well-defined even at points where g_P is not differentiable, i.e. where P has atoms. Only the equality $R_P = \nabla g_P$ requires P to be non-atomic.

In the univariate case, letting $F_P : x \mapsto P\big[(-\infty, x]\big]$ denote the usual univariate cdf of P, we have

$$\forall\, x \in \mathbb{R}^d, \qquad R_P(x) = \int_{\mathbb{R}} \operatorname{sign}(x - z) \, dP(z) = 2F_P(x) - 1.$$

Consequently, Q_P and R_P are indeed multivariate extensions of the univariate quantile map and cdf, respectively.

As we mentioned earlier, the conceptual and computational simplicity of geometric ranks and quantiles allow for explicit qualitative and quantitative results. Therefore, geometric ranks and quantiles are well-understood; see, e.g. [17] and [18] for interesting features of geometric quantiles. Similarly to their univariate counterpart, it is well-know that geometric ranks characterise probability measures in arbitrary dimension d : if P and Q are Borel probability measures on $\mathbb{R}^d$, and if $R_P(x) = R_Q(x)$ for any $x \in \mathbb{R}^d$, then $P = Q$ (see Theorem 2.5 in [10]). Note that this very desirable property is also shared by ranks based on optimal transport and, when P admits a sufficiently smooth density, the density can be recovered from the rank via a (highly non-linear) partial differential equation, see [7]. The characterisation property is not shared by the concept of halfspace depth (see [19]). However, halfspace depth possesses the characterisation property within some classes of probability measures; see [20], who gave the first positive result for empirical probability measures by algorithmically reconstructing the measure. We refer the reader to [21] for a review on the question of characterisation for halfspace depth. Therefore, it is most natural to explore the possibility of recovering a probability measure from its geometric rank. In this paper, we show that any Borel probability measure P on $\mathbb{R}^d$ can be reconstructed from its geometric rank R_P through a (potentially fractional) linear partial differential equation involving R_P only. We further show that this result holds even when P admits no density; this extends the characterisation, given by Theorem 2.5 in [10], with a degree of generality that outperforms similar results known for halfspace-depth and quantiles based on optimal transport.

The structure of this paper is as follows. Section 2 is devoted to a brief review on the analytical tools we need to state and prove the main results of this paper : distribution theory, Sobolev spaces, and fractional Laplacians. In Section 3, we prove that any probability measure P on $\mathbb{R}^d$ is related to its geometric rank through a (potentially fractional) linear PDE in the sense of distributions. We thoroughly investigate the regularity properties of geometric ranks, and give sufficient conditions for which the PDE holds pointwise. We give examples of the reconstruction procedure of probability measures supported in $\mathbb{R}^2$ and $\mathbb{R}^3$ in Section 4. By exploiting the results of Section 3, we establish some regularity properties of geometric quantile contours in Section 5. In Section 6, we give a refinement of the characterisation property of geometric ranks, for odd dimensions only, before stating a partial counterpart to the non-local nature of the PDE in even dimensions. We conclude the paper by discussing some perspectives on our new results.

Notations.
Let $d \geq 1$ and $U \subset \mathbb{R}^d$ an open subset.

- $\mathbb{N} = \{0, 1, 2, \ldots\}$ is the collection of natural numbers.
- We denote the Euclidean inner product on $\mathbb{R}^d$ by $(\cdot, \cdot)$, and the induced norm by $|\cdot|$.
- We let ${}^{d-1} = \{x \in \mathbb{R}^d : (x, x) = 1\}$ stand for the unit sphere of $\mathbb{R}^d$.
- For any $x \in \mathbb{R}^d$ and $r > 0$, we let $B_r(x)$ and B_r denote the open ball centered at x with radius r and the ball centered at the origin with radius r, respectively.
- For any subset $A \subset \mathbb{R}^d$, we write $\overline{A}$ for the closure of A with respect to the usual topology of $\mathbb{R}^d$.
- $\mathbb{I}[A]$ denotes the indicator function of the condition A.
- Fix $k \in \mathbb{N}$. Then for any k-times differentiable function $u : U \to \mathbb{C}$, we let

$$\partial^\alpha u(x) := \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}}(x), \qquad \forall\, x \in U,$$

  for any $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ such that $|\alpha| := \sum_{j=1}^d \alpha_j \leq k$. By convention, we let $\partial^\alpha u := u$ if $\alpha = (0, \ldots, 0)$. In addition, we say that
  - $u \in \mathcal{C}^k(U)$ if u is k-times differentiable and such that $\partial^\alpha u$ is continuous over U for any $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq k$;
  - $u \in \mathcal{C}_b^k(U)$ if $u \in \mathcal{C}^k(U)$ and $\partial^\alpha u$ is bounded over U for any $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq k$;
  - $u \in \mathcal{C}_c^k(U)$ if $u \in \mathcal{C}^k$ has a compact support contained in U - the set $\mathcal{C}_c^\infty(U)$ of infinitely differentiable maps with compact support in U is also denoted $\mathcal{D}(U)$;
  - $u \in \mathcal{C}^{k,\alpha}(U)$ for some $\alpha \in (0, 1]$ if we have the following : $u \in \mathcal{C}^k(U)$, $\partial^\beta u$ is bounded over U for any $\beta \in \mathbb{N}^d$ with $|\beta| \leq k$, and $\partial^\beta u$ is α-Hölder continuous over U when $|\beta| = k$;
  - $u \in \mathcal{C}_0(\mathbb{R}^d)$ if u is continuous and converges to 0 at infinity;
- When V is a collection of functions $u : \mathcal{T} \to \mathbb{C}^d$ defined over a topological space $\mathcal{T}$, we let V_{loc} denote the collection of functions $u : \mathcal{T} \to \mathbb{C}^d$ such that the restriction $u_{|K}$ of u to any compact set $K \subset \mathcal{T}$ belongs to V.

- For any $u \in L^1(\mathbb{R}^d)$, we define the Fourier transform $\widehat{u}$ of u by letting
$$\widehat{u}(\xi) = \int_{\mathbb{R}^d} u(x) e^{-2i\pi(x,\xi)}\, dx, \qquad \forall\, \xi \in \mathbb{R}^d.$$
We let $\mathcal{F} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ denote the Fourier transform over $L^2(\mathbb{R}^d)$, defined as the unique continuous extension to $L^2(\mathbb{R}^d)$ of the restriction of the Fourier transform on the Schwartz class $\mathcal{S}(\mathbb{R}^d)$; see Section 2.1 for the definition of $\mathcal{S}(\mathbb{R}^d)$. We will also denote by $\mathcal{F}$ the Fourier transform acting on the space $\mathcal{S}(\mathbb{R}^d)'$ of tempered distributions on $\mathbb{R}^d$ (see Section 2.1), and acting componentwise on $\mathcal{S}^k(\mathbb{R}^d)'$ for an arbitrary $k \in \mathbb{N}$ with $k \geq 1$.

2. Review of background material

In this section, we review some tools of analysis we need to state and prove the main results of this paper : distribution theory, Sobolev spaces, and fractional Laplacians.

2.1. **Distributions.** The main reference we used for this section is [22]. Let $U \subset \mathbb{R}^d$ be an open subset. The set of infinitely differentiable functions whose support is compact and included in U is denoted in the paper by $\mathcal{C}_c^\infty(U)$. We endow $\mathcal{C}_c^\infty(U)$ with the following notion of convergence: a sequence $(\varphi_k) \subset \mathcal{C}_c^\infty(U)$ converges to $\varphi \in \mathcal{C}_c^\infty(U)$ in the space $\mathcal{C}_c^\infty(U)$ if there exists a compact subset $K \subset U$ with $\mathrm{supp}(\varphi_k) \subset K$ for any k and such that
$$\forall\, \alpha \in \mathbb{N}^d, \qquad \lim_{k\to\infty} \sup_{x\in K} |\partial^\alpha(\varphi_k - \varphi)(x)| = 0.$$

Definition 2.1. *A distribution on U is a map $T : \mathcal{C}_c^\infty(U) \to \mathbb{C}$, $\varphi \mapsto \langle T, \varphi\rangle$ which is linear and continuous with respect to the convergence on $\mathcal{C}_c^\infty(U)$. The set of all distributions on U is denoted $\mathcal{D}(U)'$.*

Examples 2.2. We list here typical examples of distributions and a few usual ways to obtain distributions from other distributions.

(1) Any function $f \in L^1_{\mathrm{loc}}(U)$ gives rise to a distribution T_f on U which we also write f by an obivous abuse of notation, by letting
$$\langle f, \varphi\rangle := \int_U f(x)\varphi(x)\, dx, \qquad \forall \varphi \in \mathcal{C}_c^\infty(U).$$
(2) Similarly, any Borel measure μ on U that is finite over compact subsets of U leads to a distribution on U by letting
$$\langle \mu, \varphi\rangle := \int_U \varphi(x)\, d\mu(x), \qquad \forall \varphi \in \mathcal{C}_c^\infty(U).$$
In particular, any Borel probability measure is a distribution on any open subset of $\mathbb{R}^d$.
(3) If $T \in \mathcal{D}(U)'$ is a distribution on U, we define its distributional derivatives $\partial^\alpha T$, $\alpha \in \mathbb{N}^d$, by letting
$$\langle \partial^\alpha T, \varphi\rangle := (-1)^{|\alpha|} \langle T, \partial^\alpha \varphi\rangle\,, \qquad \forall \varphi \in \mathcal{C}_c^\infty(U).$$
(4) For any smooth function $f \in \mathcal{C}^\infty(U)$ and distribution $T \in \mathcal{D}(U)'$ on U, we define the distribution fT by letting
$$\langle fT, \varphi\rangle := \langle T, f\varphi\rangle\,, \qquad \forall \varphi \in \mathcal{C}_c^\infty(U).$$

Distributions are stable with respect to multiplication by smooth functions, and taking derivatives. Other common operations, such as convolution and Fourier transform, do not leave the space $\mathcal{C}_c^\infty(U)$ invariant and cannot be directly defined on $\mathcal{D}(U)'$. Consequently, we need to use another class of test functions, and the corresponding new distributions, on which we can apply these operations. This is the role of tempered distributions, that rely on the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ defined as

$$\mathcal{S}(\mathbb{R}^d) = \Big\{ f \in \mathcal{C}^\infty(\mathbb{R}^d) : \sup_{x\in\mathbb{R}^d} (1+|x|)^m |\partial^\alpha f(x)| < \infty, \ \forall m \in \mathbb{N}, \ \forall \alpha \in \mathbb{N}^d \Big\}.$$

Following our definition of $\mathcal{C}^\infty(\mathbb{R}^d)$, functions from $\mathcal{S}(\mathbb{R}^d)$ are complex-valued (see Section 1).

The set $\mathcal{S}(\mathbb{R}^d)$ is a vector space. It is also stable by scalar multiplication, multiplication by smooth functions all derivatives of which have at most polynomial growth at infinity, convolution, differentiation and Fourier transform. We further have the inclusion

$$\mathcal{C}_c^\infty(U) \subset \mathcal{S}(\mathbb{R}^d)$$

for any open subset $U \subset \mathbb{R}^d$. The set $\mathcal{S}(\mathbb{R}^d, \mathbb{C}^k)$, with $k \geq 1$, will stand for the collection of vector fields $\Psi = (\psi_1, \ldots, \psi_k)$ for which every component ψ_i belongs to $\mathcal{S}(\mathbb{R}^d)$. Similarly to the space $\mathcal{C}_c^\infty(\mathbb{R}^d)$, we endow $\mathcal{S}(\mathbb{R}^d)$ with an adequate notion of convergence. A sequence $(\psi_k) \subset \mathcal{S}(\mathbb{R}^d)$ converges to $\psi \in \mathcal{S}(\mathbb{R}^d)$ in the space $\mathcal{S}(\mathbb{R}^d)$ if

$$\forall\, m \in \mathbb{N}, \ \forall\, \alpha \in \mathbb{N}^d, \qquad \lim_{k\to\infty} \sup_{x\in\mathbb{R}^d} |(1+|x|)^m \partial^\alpha(\psi_k - \psi)(x)| = 0.$$

Definition 2.3. *A tempered distribution is a map* $T : \mathcal{S}(\mathbb{R}^d) \to \mathbb{C}, \ \psi \mapsto \langle T, \psi \rangle$ *which is linear and continuous with respect to the convergence on* $\mathcal{S}(\mathbb{R}^d)$. *The set of tempered distributions is denoted* $\mathcal{S}(\mathbb{R}^d)'$.

For the sake of simplicity, for any $k \geq 1$ we let $\mathcal{S}^k(\mathbb{R}^d)' := (\mathcal{S}(\mathbb{R}^d)')^k$ denote the set of linear maps

$$T = (T_1, \ldots, T_k) : \mathcal{S}(\mathbb{R}^d, \mathbb{C}^k) \to \mathbb{C}^k, \ \Psi = (\psi_1, \ldots, \psi_k) \mapsto (\langle T_1, \psi_1 \rangle, \ldots, \langle T_k, \psi_k \rangle)$$

such that $T_i \in \mathcal{S}(\mathbb{R}^d)'$ for any $i \in \{1, \ldots, k\}$. We let all operations described above act on $\mathcal{S}^k(\mathbb{R}^d)'$ componentwise. Therefore, the identities we stated remain valid on $\mathcal{S}^k(\mathbb{R}^d)'$.

It is easy to see that if a sequence $(\varphi_k) \subset \mathcal{C}_c^\infty(\mathbb{R}^d)$ converges to some $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ in the space $\mathcal{C}_c^\infty(\mathbb{R}^d)$, then convergence also holds in the space $\mathcal{S}(\mathbb{R}^d)'$. In particular, any tempered distribution is a distribution over $\mathbb{R}^d$.

Examples 2.4. Below, we list typical examples of tempered distributions, and a few usual ways to obtain tempered distributions from other tempered distributions.

(1) If $f : \mathbb{R}^d \to \mathbb{C}$ is a measurable function such that, for some $m \in \mathbb{N}$ and $p \in [1, \infty)$, the map $x \mapsto f(x)(1+|x|)^{-m}$ belongs to $L^p(\mathbb{R}^d)$, then $f \in \mathcal{S}(\mathbb{R}^d)'$ by letting

$$\langle f, \psi \rangle := \int_{\mathbb{R}^d} f(x)\psi(x)\, dx, \qquad \forall\, \psi \in \mathcal{S}(\mathbb{R}^d).$$

(2) Similarly, if μ is a Borel measure on $\mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} \frac{1}{(1+|x|)^m}\, d\mu(x) < \infty$$

for some $m \in \mathbb{N}$, then $\mu \in \mathcal{S}(\mathbb{R}^d)'$.

(3) Let $T \in \mathcal{S}(\mathbb{R}^d)'$ and $f \in \mathcal{C}^\infty(\mathbb{R}^d)$. We have already mentioned that the product fT is a distribution over $\mathbb{R}^d$. For fT to be tempered, we need $f\psi$ to be a Schwartz function for any $\psi \in \mathcal{S}(\mathbb{R}^d)$. This will be the case, for instance, if f and all its derivatives have at most polynomial growth at infinity. If no restriction is imposed on the growth of f and its derivatives, the product fT might not be tempered; consider, e.g., the tempered distribution $T \equiv 1$, and the smooth function $f(x) = e^x$.

(4) If $T \in \mathcal{S}(\mathbb{R}^d)'$ is a tempered distribution, then $\partial^\alpha T$ is also a tempered distribution for any $\alpha \in \mathbb{N}^d$.

(5) If $T \in \mathcal{S}(\mathbb{R}^d)'$, we define its Fourier transform $\mathcal{F}T$ by letting

$$\langle \mathcal{F}T, \psi \rangle = \langle T, \widehat{\psi} \rangle, \qquad \forall\, \psi \in \mathcal{S}(\mathbb{R}^d).$$

The map $\mathcal{F}T$ is a tempered distribution. Just as for smooth functions over $\mathbb{R}^d$, the equalities

$$\mathcal{F}(\partial^\alpha T) = (2i\pi\xi)^\alpha \mathcal{F}(T), \qquad \text{and} \qquad \partial^\alpha \mathcal{F}(T) = \mathcal{F}((-2i\pi x)^\alpha T)$$

hold in $\mathcal{S}(\mathbb{R}^d)'$ for any $\alpha \in \mathbb{N}^d$.

Proposition 2.5. *Let $T \in \mathcal{S}(\mathbb{R}^d)'$ and $\psi \in \mathcal{S}(\mathbb{R}^d)$. We define the convolution $T * \psi : \mathbb{R}^d \to \mathbb{C}$ by letting*

$$(T * \psi)(x) = \langle T, \psi(x - \cdot) \rangle, \qquad \forall\, x \in \mathbb{R}^d.$$

The map $T * \psi$ belongs to $\mathcal{C}^\infty(\mathbb{R}^d)$, and we have

$$\forall\, \alpha \in \mathbb{N}^d, \qquad \partial^\alpha (T * \psi) = (\partial^\alpha T) * \psi = T * (\partial^\alpha \psi)$$

over $\mathbb{R}^d$. Furthermore, $T * \psi$ has polynomial growth. In particular, $T * \psi$ is a tempered distribution over $\mathbb{R}^d$, and the equality

$$\mathcal{F}(T * \psi) = \mathcal{F}(T)\, \widehat{\psi}$$

holds in $\mathcal{S}(\mathbb{R}^d)'$.

2.2. **Sobolev spaces.** The main reference we used for this section is [23]. Let $U \subset \mathbb{R}^d$ be an open subset. A function $u \in L^1_{\text{loc}}(U)$ has weak derivatives of order k in U if, for any $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq k$, the distributional derivative $\partial^\alpha u$ is actually a function, and belongs to $L^1_{\text{loc}}(U)$, i.e. there exists $v_\alpha \in L^1_{\text{loc}}(U)$ such that

$$\forall\, \varphi \in \mathcal{C}_c^\infty(U), \qquad \int_U u(x) \partial^\alpha \varphi(x)\, dx = (-1)^{|\alpha|} \int_U v_\alpha(x) \varphi(x)\, dx$$

We write $v_\alpha = \partial^\alpha u$. When $u \in \mathcal{C}^k(U)$, then $\partial^\alpha u$ coincides with the usual partial derivative of u. In the sequel, when no regularity of $u \in L^1_{\text{loc}}(U)$ is assumed, then $\partial^\alpha u$ will always stand for a distributional derivative.

For any integer $k \geq 1$, we define the Sobolev space

$$H^k(U) = \Big\{ u \in L^2(U) : \partial^\alpha u \in L^2(U),\ \forall \alpha \in \mathbb{N}^d, |\alpha| \leq k \Big\}.$$

Since $\mathcal{F}(\partial^\alpha u) = (2i\pi x)^\alpha \mathcal{F}u$ in $\mathcal{S}(\mathbb{R}^d)'$, and since a function belongs to $L^2(\mathbb{R}^d)$ if and only if its distributional Fourier transform does, the condition "$u \in L^2(\mathbb{R}^d)$ and $\partial^\alpha u \in L^2(\mathbb{R}^d)$"

is equivalent to "$u \in L^2(\mathbb{R}^d)$ and $x^\alpha \mathcal{F}u \in L^2(\mathbb{R}^d)$". It follows that we can, equivalently, define $H^k(\mathbb{R}^d)$ as

$$H^k(\mathbb{R}^d) = \Big\{ u \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} (1 + |\xi|^{2k}) |\mathcal{F}u(\xi)|^2 \, d\xi < \infty \Big\}.$$

For any real $s > 0$, we finally let

$$\begin{aligned} H^s(\mathbb{R}^d) &= \Big\{ u \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} (1 + |\xi|^{2s}) |\mathcal{F}u(\xi)|^2 \, d\xi < \infty \Big\} \\ &= \Big\{ u \in L^2(\mathbb{R}^d) : (1 + |\xi|^2)^{s/2} \mathcal{F}u(\xi) \in L^2(\mathbb{R}^d) \Big\}. \end{aligned}$$

For any $s > 0$, the set $H^s(\mathbb{R}^d)$ is a Hilbert space, equipped with the inner product

$$(u, v)_{H^s(\mathbb{R}^d)} = \int_{\mathbb{R}^d} (1 + |\xi|^2)^s \mathcal{F}u(\xi) \overline{\mathcal{F}v(\xi)} \, d\xi.$$

Sobolev spaces are particularly approriate to study the regularity of distributional solutions u to the Laplace equation $-\Delta u = f$ when u and f satisfy mild assumptions, described in the next definition.

Definition 2.6. *Let $d \geq 1$, and $\Omega \subset \mathbb{R}^d$ be an open and bounded set. Let $g \in L^2(\Omega)$, and $u \in H^1(\Omega)$. We say that u satisfies $-\Delta u = f$ in the weak sense in Ω if*

$$\forall \, \varphi \in \mathcal{C}_c^\infty(\Omega), \qquad \int_\Omega \langle \nabla u(x), \nabla \varphi(x) \rangle \, dx = \int_\Omega f(x) \varphi(x) \, dx.$$

The following proposition is Corollary 2.17 in [24]. It will play an important role in our proofs.

Proposition 2.7 (Elliptic regularity)**.** *Let $d \geq 1$ and B_1 the open unit ball of $\mathbb{R}^d$. Fix $\alpha \in (0, 1)$, $k \in \mathbb{N}$, and $f \in \mathcal{C}^{k,\alpha}(B_1)$. If $u \in H^1(B_1) \cap L^\infty(B_1)$ satisfies $-\Delta u = f$ in the weak sense in B_1, then $u \in \mathcal{C}^{k+2,\alpha}_{loc}(B_1)$.*

We will use a straightforward generalization of Proposition 2.7, stated in the next corollary. For the sake of completness, we prove it in Appendix A.1

Corollary 2.8. *Let $d \geq 1$ and $\Omega \subset \mathbb{R}^d$ an open subset. Fix $\alpha \in (0, 1)$, $k \in \mathbb{N}$, and $f \in \mathcal{C}^{k,\alpha}_{loc}(\Omega)$. If $u \in H^1_{loc}(\Omega) \cap L^\infty_{loc}(\Omega)$ satifies $-\Delta u = f$ in the weak sense in Ω, then $u \in \mathcal{C}^{k+2,\alpha}_{loc}(\Omega)$.*

2.3. **Basics of fractional Laplacians.** Different definitions of fractional Laplacians exist. Some rely on Fourier transform, others on singular integrals, or Sobolev spaces. They all coincide for functions with enough regularity, such as the Schwartz class, but may differ in general, or at least be defined over different domains. In this section, we provide a self-contained introduction to fractional Laplacians. The approach we present is based on Fourier transforms, because it appears under this form in our proofs. The main references we used for this section are [25], [26], [27], and [28].

Let us fix $u \in \mathcal{S}(\mathbb{R}^d)$. Recalling that $\mathcal{F}((-\Delta)^\ell u) = (2\pi|\xi|)^{2\ell} \mathcal{F}u$ for any integer $\ell \geq 0$, we let

$$\forall \, s \in (0, \infty), \; \forall \, x \in \mathbb{R}^d, \qquad ((-\Delta)^s u)(x) := (2\pi)^{2s} \mathcal{F}^{-1}(|\xi|^{2s} \mathcal{F}u(\xi))(x).$$

We now make a comment on the factor $(2\pi)^{2s}$ in the definition of $(-\Delta)^s u$. Obivously, it is a consequence of our choice of normalization in the definition of the Fourier transform. For another normalization in the Fourier transform, $\mathcal{F}_{a,b}$ say, defined by

$$(\mathcal{F}_{a,b}u)(\xi) := \frac{1}{b}\int_{\mathbb{R}^d} u(x)e^{-ia(x,\xi)}\,dx, \qquad \forall\ \xi \in \mathbb{R}^d,$$

for some $a > 0$ and $b > 0$, we have

$$\forall\ \xi \in \mathbb{R}^d, \qquad (\mathcal{F}_{a,b}^{-1}u)(\xi) = b\Big(\frac{a}{2\pi}\Big)^d \int_{\mathbb{R}^d} u(x)e^{ia(x,\xi)}\,dx.$$

It is easy to show that

$$\forall\ x \in \mathbb{R}^d, \qquad a^{2s}\mathcal{F}_{a,b}^{-1}(|\xi|^{2s}\mathcal{F}_{a,b}u(\xi))(x) = \mathcal{F}_{1,1}^{-1}(|\xi|^{2s}\mathcal{F}_{1,1}u(\xi))(x).$$

It follows that any choice of a and b leads to the same value of $(-\Delta)^s u$ if we let

$$\forall\ x \in \mathbb{R}^d, \qquad ((-\Delta)^s u)(x) = a^{2s}\mathcal{F}_{a,b}^{-1}(|\xi|^{2s}\mathcal{F}_{a,b}u(\xi))(x).$$

In the sequel, we will be working with $a = 2\pi$ and $b = 1$.

When $s = n+\sigma$, with $n \in \mathbb{N}$ and $\sigma \in (0,1)$, taking the Fourier transform readily implies that

$$(-\Delta)^s u = (-\Delta)^\sigma((-\Delta)^n u),$$

where $(-\Delta)^n$ is the usual differential operator $-\Delta$ taken n times. Let us therefore restrict to $s \in (0,1)$. It is proved in [26] (see Proposition 3.3) that, in this case, we have

$$\forall\ x \in \mathbb{R}^d, \qquad ((-\Delta)^s u)(x) = c_{d,s} \lim_{\eta\to 0}\int_{\mathbb{R}^d\setminus B_\eta(x)} \frac{u(x)-u(z)}{|x-z|^{d+2s}}\,dz, \tag{2.1}$$

for some constant $c_{d,s}$ that only depends on d and s. Note that the normalization of the Fourier transform used in [26] corresponds to $a = 1$ and $b = (2\pi)^{\frac{d}{2}}$ in our previous discussion. The value of the constant $c_{d,s}$ can be found in [28] (see Theorem 1), and is given by

$$c_{d,s} = \frac{s(1-s)4^s\Gamma(d/2+s)}{|\Gamma(2-s)|\pi^{d/2}}. \tag{2.2}$$

We will now explain how one can extend the domain of $(-\Delta)^s$. It is easy to see that

$$\forall\ u, v \in \mathcal{S}(\mathbb{R}^d), \qquad \int_{\mathbb{R}^d}((-\Delta)^s u)(x)v(x)\,dx = \int_{\mathbb{R}^d} u(x)((-\Delta)^s v)(x)\,dx.$$

Consequently, it is tempting to define the fractional Laplacian $(-\Delta)^s T$ of an arbitrary tempered distribution $T \in \mathcal{S}(\mathbb{R}^d)'$ (recall that Fourier transforms are involved in the definition of $(-\Delta)^s$) by letting

$$\forall\ \psi \in \mathcal{S}(\mathbb{R}^d), \qquad \langle(-\Delta)^s T, \psi\rangle := \langle T, (-\Delta)^s\psi\rangle\,. \tag{2.3}$$

However, this approach must be discarded because $(-\Delta)^s\psi$ does not belong to $\mathcal{S}(\mathbb{R}^d)$ in general. The regularity of $(-\Delta)^s\psi$ is established in the next proposition, which is stated in [27] but not proved. For the sake of completness, we provide a proof of this proposition in Appendix A.2.

Proposition 2.9. *Let $d \geq 1$, $s \in (0,1)$, and $u \in \mathcal{S}(\mathbb{R}^d)$. Then $(-\Delta)^s u \in \mathcal{C}^\infty(\mathbb{R}^d)$ and*

$$\forall\ \alpha \in \mathbb{N}^d, \qquad \sup_{x\in\mathbb{R}^d} |(1+|x|^{d+2s})\partial^\alpha((-\Delta)^s u)(x)| < \infty. \tag{2.4}$$

In addition, for any $\alpha \in \mathbb{N}^d$ we have

$$\sup_{x\in\mathbb{R}^d} |(1+|x|^{d+2s})\partial^\alpha((-\Delta)^s u)(x)| \lesssim |\partial^\alpha u|_{L^1(\mathbb{R}^d)} + \sup_{z\in\mathbb{R}^d} \Big((1+|z|)^{d+2}|\nabla^2(\partial^\alpha u)(z)|\Big),$$

where, for any smooth function ψ, we let $|\nabla^2\psi(z)|$ stand for the operator norm of the Hessian matrix $\nabla^2\psi(z)$ of ψ at z.

According to Proposition 2.9, the space of test functions

$$\mathcal{S}_s(\mathbb{R}^d) := \Big\{\psi \in \mathcal{C}^\infty(\mathbb{R}^d) : \sup_{x\in\mathbb{R}^d} |(1+|x|^{d+2s})\partial^\alpha\psi(x)| < \infty,\ \forall\alpha \in \mathbb{N}^d\Big\}$$

is more appropriate to define fractional Laplacians by duality, as we will explain below. For any $k \geq 1$, we denote $\mathcal{S}_s(\mathbb{R}^d, \mathbb{C}^k)$ the collection of vector fields $\Psi = (\psi_1, \ldots, \psi_k)$ for which $\psi_i \in \mathcal{S}_s(\mathbb{R}^d)$ for any $i = 1, \ldots, k$. Similarly to tempered distributions, we let $\mathcal{S}_s^k(\mathbb{R}^d)'$ stand the space $(\mathcal{S}_s(\mathbb{R}^d)')^k$ for any $k \in \mathbb{N}$ with $k \geq 1$; see Definition 2.3 and the comments below. We endow $\mathcal{S}_s(\mathbb{R}^d)$ with the following convergence: a sequence $(\psi_k) \subset \mathcal{S}_s(\mathbb{R}^d)$ converges to $\psi \in \mathcal{S}_s(\mathbb{R}^d)$ in the space $\mathcal{S}_s(\mathbb{R}^d)$ if

$$\forall\ \alpha \in \mathbb{N}^d, \qquad \lim_{k\to\infty} \sup_{x\in\mathbb{R}^d} |(1+|x|^{d+2s})\partial^\alpha(\psi_k - \psi)(x)| = 0.$$

Definition 2.10. *Fix $s \in (0,1)$. We let $\mathcal{S}_s(\mathbb{R}^d)'$ be the set of maps $T : \mathcal{S}_s(\mathbb{R}^d) \to \mathbb{C}$, $\psi \mapsto \langle T, \psi\rangle$ which are linear and continuous with respect to the convergence on $\mathcal{S}_s(\mathbb{R}^d)$.*

Proposition 2.9 entails that if a sequence $(\psi_k) \subset \mathcal{S}(\mathbb{R}^d)$ converges to 0 in the space $\mathcal{S}(\mathbb{R}^d)$, then the sequence $((-\Delta)^s\psi_k)$ converges to 0 in the space $\mathcal{S}_s(\mathbb{R}^d)$ as $k \to \infty$. Therefore, $\mathcal{S}(\mathbb{R}^d)$ continuously embeds into $\mathcal{S}_s(\mathbb{R}^d)$. In particular, if $(-\Delta)^s T$ is defined according to (2.3) then $(-\Delta)^s T$ is a tempered distribution, provided $T \in \mathcal{S}_s(\mathbb{R}^d)'$.

Because $\mathcal{S}(\mathbb{R}^d)$ continuously embeds into $\mathcal{S}_s(\mathbb{R}^d)$, we have $\mathcal{S}_s(\mathbb{R}^d)' \subset \mathcal{S}(\mathbb{R}^d)'$. In particular, distributions in $\mathcal{S}_s(\mathbb{R}^d)'$ are more regular than those from $\mathcal{S}(\mathbb{R}^d)'$.

Definition 2.11. *Let $s \in (0,1)$. For any $T \in \mathcal{S}_s(\mathbb{R}^d)'$, we let $(-\Delta)^s T : \mathcal{S}(\mathbb{R}^d) \to \mathbb{C}$ be the linear map defined by*

$$\langle(-\Delta)^s T, \psi\rangle := \langle T, (-\Delta)^s\psi\rangle\,, \qquad \forall\ \psi \in \mathcal{S}(\mathbb{R}^d).$$

The map $(-\Delta)^s T$ is a tempered distribution on $\mathbb{R}^d$.

The class $\mathcal{S}_s(\mathbb{R}^d)$ is obviously closed under differentiation. However, it is not closed under Fourier transform. Consequently, the space $\mathcal{S}_s(\mathbb{R}^d)'$ is closed under differentiation, but not under Fourier transform. In addition, it is easy to see that

$$\forall\ \alpha \in \mathbb{N}^d,\ \forall\ u \in \mathcal{S}(\mathbb{R}^d), \qquad \partial^\alpha(-\Delta)^s u = (-\Delta)^s\partial^\alpha u.$$

This implies that

$$\forall\ \alpha \in \mathbb{N}^d,\ \forall\ T \in \mathcal{S}_s(\mathbb{R}^d)', \qquad \partial^\alpha(-\Delta)^s T = (-\Delta)^s\partial^\alpha T.$$

Let us now give examples of tempered distributions that also belong to $\mathcal{S}_s(\mathbb{R}^d)'$. It is trivial to see that any (signed) measure μ such that

$$\int_{\mathbb{R}^d} \frac{1}{1+|x|^{d+2s}}\, d|\mu|(x) < \infty$$

belongs to $\mathcal{S}_s(\mathbb{R}^d)'$. In particular, any Borel probability measure on $\mathbb{R}^d$ belongs to $\mathcal{S}_s(\mathbb{R}^d)'$. This also entails that any function $u : \mathbb{R}^d \to \mathbb{C}$ belongs to $\mathcal{S}_s(\mathbb{R}^d)'$, provided that

$$\int_{\mathbb{R}^d} \frac{|u(x)|}{1+|x|^{d+2s}}\, dx < \infty.$$

In particular, we have $L^p(\mathbb{R}^d) \subset \mathcal{S}_s(\mathbb{R}^d)'$ for any $p \in [1, +\infty]$.

We now provide a result allowing one to compute $(-\Delta)^s u$ explicitly in different cases. They are not new, and proved under various assumptions in the litterature according to the definition used for the fractional Laplacian. For the sake of clarity and completness, we prove the next result in Appendix A.2.

Proposition 2.12. *Let $s \in (0,1)$, and $u \in \mathcal{S}_s(\mathbb{R}^d)'$. Then, we have the following :*

(1) If $u \in H^{2s}(\mathbb{R}^d)$, then $(-\Delta)^s u \in L^2(\mathbb{R}^d)$, and we have

$$(-\Delta)^s u = (2\pi)^{2s} \mathcal{F}^{-1}(|\xi|^{2s} \mathcal{F} u)$$

in $L^2(\mathbb{R}^d)$;

(2) If $\mathcal{F}u \in L^1_{loc}(\mathbb{R}^d)$ and $|\xi|^{2s}\mathcal{F}u(\xi) \in L^1(\mathbb{R}^d)$, then $(-\Delta)^s u \in \mathcal{C}_0(\mathbb{R}^d)$; in this case, we have

$$\forall\, x \in \mathbb{R}^d, \qquad ((-\Delta)^s u)(x) = (2\pi)^{2s} \mathcal{F}^{-1}(|\xi|^{2s} \mathcal{F} u)(x);$$

(3) If, for some open subset $\Omega \subset \mathbb{R}^d$ and $\alpha \in (0, 2-2s)$, we have $u \in \mathcal{C}^{k,\beta}(\Omega)$ with $k = \lfloor 2s+\alpha \rfloor \in \{0,1\}$ and $\beta = 2s + \alpha - k \in [0,1)$, then $(-\Delta)^s u \in \mathcal{C}^0(\Omega)$; in this case, we have

$$\forall\, x \in \Omega, \qquad (-\Delta)^s(x) = c_{d,s} \lim_{\eta \to 0} \int_{\mathbb{R}^d \setminus B_\eta(x)} \frac{u(x) - u(z)}{|x-z|^{d+2s}}\, dz.$$

3. Recovery of a probability measure from its geometric rank

The key ingredient to recover P from R_P consists in noticing that R_P writes as the convolution between P and a fixed kernel $K : \mathbb{R}^d \to \mathbb{R}^d$, defined as

$$K(x) = \begin{cases} \frac{x}{|x|} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

When confusion about the dimension is possible, we will write K_d instead of K to emphasize that K is defined over $\mathbb{R}^d$. Now, the heuristics to solve the problem is straightforward: formally taking the Fourier transform $\mathcal{F}(R_P)$ of R_P gives

$$\mathcal{F}(R_P) = \mathcal{F}(K)\mathcal{F}(P). \tag{3.1}$$

This fact was already noticed in [10], and we used this idea as a building block to prove the results of this section. Recovering P then essentially amounts to isolating $\mathcal{F}(P)$ in (3.1), and taking the inverse Fourier transform of $\mathcal{F}(P)$.

To do so, let us introduce the operator $\mathcal{L}_d$ and its adjoint $\mathcal{L}_d^*$, that will play a key role. They involve a constant γ_d, defined as

$$\frac{1}{\gamma_d} = 2^d \pi^{\frac{d-1}{2}} \Gamma\Big(\frac{d+1}{2}\Big).$$

The operators $\mathcal{L}_d$ and $\mathcal{L}_d^*$ depend on d, and so do their domains $D(\mathcal{L}_d)$ and $D(\mathcal{L}_d^*)$. We let

$$D(\mathcal{L}_d) = \begin{cases} \mathcal{S}^d(\mathbb{R}^d)' & \text{if } d \text{ is odd,} \\ \mathcal{S}^d_{1/2}(\mathbb{R}^d)' & \text{if } d \text{ is even,} \end{cases} \quad \text{and} \quad D(\mathcal{L}_d^*) = \begin{cases} \mathcal{S}(\mathbb{R}^d)' & \text{if } d \text{ is odd,} \\ \mathcal{S}_{1/2}(\mathbb{R}^d)' & \text{if } d \text{ is even.} \end{cases}$$

Recall that $\mathcal{S}(\mathbb{R}^d)'$ (resp. $\mathcal{S}^d(\mathbb{R}^d)'$) is the space of $\mathbb{C}$ (resp. $\mathbb{C}^d$)-valued tempered distributions (see Section 2.1); see Section 2.3 for the definition of $\mathcal{S}_{1/2}(\mathbb{R}^d)'$ and $\mathcal{S}^d_{1/2}(\mathbb{R}^d)'$. In particular, we have

$$D(\mathcal{L}_d) \subset \mathcal{S}^d(\mathbb{R}^d)'$$

for any d. Also notice that, because $L^\infty(\mathbb{R}^d, \mathbb{R}^d) \subset \mathcal{S}^d_{1/2}(\mathbb{R}^d)'$ and R_P belongs to $L^\infty(\mathbb{R}^d, \mathbb{R}^d)$, then $R_P \in D(\mathcal{L}_d)$ for any d.

Definition 3.1. *Let $d \in \mathbb{N}$ with $d \geq 1$. Define the (potentially fractional) differential operator $\mathcal{L}_d : D(\mathcal{L}_d) \subset \mathcal{S}^d(\mathbb{R}^d)' \to \mathcal{S}(\mathbb{R}^d)'$ by letting*

$$\mathcal{L}_d := \gamma_d \begin{cases} (-\Delta)^{\frac{d-1}{2}} \nabla \cdot & \text{if } d \text{ is odd,} \\ (-\Delta)^{\frac{1}{2}} (-\Delta)^{\frac{d-2}{2}} \nabla \cdot & \text{if } d \text{ is even,} \end{cases}$$

where $\nabla \cdot$ is the divergence operator, $(-\Delta)^k$ stands for the Laplacian operator $-\Delta$ taken k times successively when $k \in \mathbb{N}$, and $(-\Delta)^{\frac{1}{2}}$ denotes the fractional Laplacian introduced in Section 2.3.

Define the formal adjoint $\mathcal{L}_d^* : D(\mathcal{L}_d^*) \subset \mathcal{S}(\mathbb{R}^d)' \to \mathcal{S}^d(\mathbb{R}^d)'$ of $\mathcal{L}_d$ by

$$\mathcal{L}_d^* := \gamma_d \begin{cases} \nabla(-\Delta)^{\frac{d-1}{2}} & \text{if } d \text{ is odd,} \\ \nabla(-\Delta)^{\frac{1}{2}} (-\Delta)^{\frac{d-2}{2}} & \text{if } d \text{ is even,} \end{cases}$$

where ∇ stands for the gradient operator.

We call $\mathcal{L}_d^*$ the formal adjoint of $\mathcal{L}_d$ because, letting $\langle \cdot, \cdot \rangle$ denote the distributional bracket (see Definition 2.1), we have

$$\forall\, \Lambda \in D(\mathcal{L}_d),\ \forall\, \varphi \in \mathcal{S}(\mathbb{R}^d), \qquad \langle \mathcal{L}_d \Lambda, \varphi \rangle = \langle \Lambda, \mathcal{L}_d^* \varphi \rangle\,,$$

and

$$\forall\, T \in D(\mathcal{L}_d^*),\ \forall\, \Psi \in \mathcal{S}(\mathbb{R}^d, \mathbb{C}^d), \qquad \langle \mathcal{L}_d^* T, \Psi \rangle = \langle T, \mathcal{L}_d \Psi \rangle\,.$$

In particular, we have

$$\forall\, \Psi \in \mathcal{S}(\mathbb{R}^d, \mathbb{C}^d),\ \forall\, \varphi \in \mathcal{S}(\mathbb{R}^d), \qquad \int_{\mathbb{R}^d} (\mathcal{L}_d \Psi)(x) \varphi(x)\, dx = \int_{\mathbb{R}^d} \Big(\Psi(x), (\mathcal{L}_d^* \varphi)(x) \Big)\, dx.$$

Taking Fourier transforms shows that all differential operators involved in the definition of $\mathcal{L}_d$ and $\mathcal{L}_d^*$ commute over $D(\mathcal{L}_d)$ and $D(\mathcal{L}_d^*)$, respectively. This legitimates writing $\mathcal{L}_d$ and $\mathcal{L}_d^*$ in the more compact forms

$$\mathcal{L}_d = \gamma_d\, (-\Delta)^{\frac{d-1}{2}} \nabla \cdot\,, \quad \text{and} \quad \mathcal{L}_d^* = \gamma_d\, \nabla (-\Delta)^{\frac{d-1}{2}},$$

irrespective of $d \geq 1$. In $\mathbb{R}$, notice that $\mathcal{L}_1$ and $\mathcal{L}_1^*$ simply reduce to

$$\mathcal{L}_1 = \frac{1}{2} \frac{d}{dx} = \mathcal{L}_1^*.$$

3.1. **Distributional recovery.** In Theorem 3.2, we assume that P admits a well-behaved density and explicitly recover the density from R_P. Then, we extend this result to an arbitrary probability measure P in Theorem 3.3, and recover P from R_P in the sense of distributions.

Theorem 3.2. *Let $d \geq 1$ and P a Borel probability measure on $\mathbb{R}^d$. Assume that P admits a density $f_P \in \mathcal{S}(\mathbb{R}^d)$ with respect to the Lebesgue measure. Then $R_P \in \mathcal{C}^\infty(\mathbb{R}^d)$ and we have $f_P(x) = (\mathcal{L}_d R_P)(x)$ for any $x \in \mathbb{R}^d$.*

Proof of Theorem 3.2. Recall that $R_P(x) = (K * f_P)(x)$ for any $x \in \mathbb{R}^d$, where K is the kernel introduced at the beginning of Section 3. Because $K \in \mathcal{S}^d(\mathbb{R}^d)'$ and $f_P \in \mathcal{S}(\mathbb{R}^d)$, Proposition 2.5 entails that $R_P \in \mathcal{C}^\infty(\mathbb{R}^d)$. When $d = 1$ recall that $R_P = 2F_P - 1$, where

$$F_P(x) = \int_{-\infty}^{x} f_P(t)\, dt$$

is the cdf of P; see Section 1. Because f_P is continuous, the fundamental theorem of calculus yields

$$(\mathcal{L}_d R_P)(x) = \gamma_d\, (-\Delta)^{\frac{d-1}{2}} (\nabla \cdot R_P)(x) = \frac{1}{2}\frac{dR_P}{dx}(x) = \frac{1}{2}\frac{d(2F_P - 1)}{dx}(x) = f_P(x).$$

Therefore, the claim is proved when $d = 1$.

Assume that $d \geq 2$. Because $K \in L^\infty(\mathbb{R}^d, \mathbb{R}^d)$, we have $K \in \mathcal{S}^d(\mathbb{R}^d)'$. Therefore, Proposition 2.5 entails that the equality

$$\mathcal{F}(R_P) = \mathcal{F}(K)\, \widehat{f_P} \tag{3.2}$$

holds in $\mathcal{S}^d(\mathbb{R}^d)'$, because $f_P \in \mathcal{S}(\mathbb{R}^d)$. Lemma A.1 and the fact that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ yield

$$\mathcal{F}\Big(\frac{1}{|x|}\Big)(\xi) = \frac{\Gamma(\frac{d-1}{2})}{\pi^{\frac{d-1}{2}}} \frac{1}{|\xi|^{d-1}}$$

in $\mathcal{S}(\mathbb{R}^d)'$; recall that $1/|x|$ is a tempered distribution on $\mathbb{R}^d$ because $d \geq 2$. From the identities stated before Proposition 2.5, we deduce that

$$\mathcal{F}(K) = \mathcal{F}\Big(\frac{x}{|x|}\Big) = -\frac{1}{2i\pi}\nabla\mathcal{F}\Big(\frac{1}{|x|}\Big) = -\frac{1}{2i\pi}\frac{\Gamma(\frac{d-1}{2})}{\pi^{\frac{d-1}{2}}}\nabla\Big(\frac{1}{|\xi|^{d-1}}\Big)$$

in $\mathcal{S}^d(\mathbb{R}^d)'$. Recalling that $x\Gamma(x) = \Gamma(x+1)$ for any $x > 0$, Lemma A.3 yields

$$(\mathcal{F}K)(\xi) = \frac{\Gamma(\frac{d+1}{2})}{i\pi^{\frac{d+1}{2}}}\ \mathrm{P.V.}\Big(\frac{\xi}{|\xi|^{d+1}}\Big) \tag{3.3}$$

in $\mathcal{S}^d(\mathbb{R}^d)'$. Equation (3.2) then rewrites

$$(\mathcal{F}R_P)(\xi) = \frac{\Gamma(\frac{d+1}{2})}{i\pi^{\frac{d+1}{2}}}\ \mathrm{P.V.}\Big(\frac{\xi}{|\xi|^{d+1}}\Big)\widehat{f_P}(\xi).$$

On the one hand, we have

$$\Big(\xi,\ \mathrm{P.V.}\big(\frac{\xi}{|\xi|^{d+1}}\big)\Big) := \sum_{i=1}^{d} \xi_i\ \mathrm{P.V.}\big(\frac{\xi_i}{|\xi|^{d+1}}\big) = \frac{1}{|\xi|^{d-1}}$$

in $\mathcal{S}(\mathbb{R}^d)'$. On the other hand, we have

$$\big(\xi,\ (\mathcal{F}R_P)(\xi)\big) = \sum_{i=1}^{d} \xi_i \mathcal{F}((R_P)_i)(\xi) = \frac{1}{2i\pi}\sum_{i=1}^{d} \mathcal{F}(\partial_i (R_P)_i)(\xi) = \frac{1}{2i\pi}\mathcal{F}(\nabla\cdot R_P)(\xi)$$

in $\mathcal{S}(\mathbb{R}^d)'$. It follows that

$$\frac{1}{2i\pi}\mathcal{F}(\nabla\cdot R_P)(\xi) = \frac{\Gamma(\frac{d+1}{2})}{i\pi^{\frac{d+1}{2}}}\frac{1}{|\xi|^{d-1}}\widehat{f_P}(\xi) \tag{3.4}$$

in $\mathcal{S}(\mathbb{R}^d)'$. Let us consider two cases. (A) Assume that $d \geq 3$ is odd. Therefore, $\frac{d-1}{2} \in \mathbb{N}$ and we have

$$\widehat{f_P}(\xi) = \frac{1}{2}\frac{\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d+1}{2})}|\xi|^{d-1}\mathcal{F}(\nabla\cdot R_P)(\xi) = \gamma_d\ \mathcal{F}((-\Delta)^{\frac{d-1}{2}}\nabla\cdot R_P)(\xi)$$

in $\mathcal{S}(\mathbb{R}^d)'$, where $\gamma_d^{-1} = 2^d\pi^{\frac{d-1}{2}}\Gamma(\frac{d+1}{2})$. In particular, the equality

$$f_P = \gamma_d\ (-\Delta)^{\frac{d-1}{2}}(\nabla\cdot R_P) = \mathcal{L}_d(R_P)$$

holds in $\mathcal{S}(\mathbb{R}^d)'$. The fact that R_P belongs to $\mathcal{C}^\infty(\mathbb{R}^d)$ ensures that the r.h.s. of the last equality is a continuous function. Because f_P is also continuous, and equality holds in the sense of distributions, equality also holds pointwise. (B) Assume that $d \geq 2$ is even. Because $d-2$ is even, we deduce from (3.4) that

$$\frac{\widehat{f_P}(\xi)}{|\xi|} = \frac{1}{2}\frac{\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d+1}{2})}|\xi|^{d-2}\mathcal{F}(\nabla\cdot R_P)(\xi) = \frac{1}{2}\frac{\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d+1}{2})}\frac{1}{(2\pi)^{d-2}}\mathcal{F}((-\Delta)^{\frac{d-2}{2}}(\nabla\cdot R_P))(\xi)$$

holds in $\mathcal{S}(\mathbb{R}^d)'$. Let us recall that $R_P \in \mathcal{S}^d_{1/2}(\mathbb{R}^d)'$. Since $\mathcal{S}^d_{1/2}(\mathbb{R}^d)'$ is closed with respect to differentiation, we have $u \in \mathcal{S}_{1/2}(\mathbb{R}^d)'$, with $u := (-\Delta)^{\frac{d-2}{2}}(\nabla\cdot R_P)$. It is clear that $\mathcal{F}u \in L^1_{\text{loc}}(\mathbb{R}^d)$ since $\widehat{f_P}(\xi)/|\xi| \in L^1_{\text{loc}}(\mathbb{R}^d)$ (recall that $d \geq 2$), and that $|\xi|\mathcal{F}u(\xi) \in L^1(\mathbb{R}^d)$ since $f_P \in \mathcal{S}(\mathbb{R}^d)$. It follows from Proposition 2.12 that $(-\Delta)^{\frac{1}{2}}(-\Delta)^{\frac{d-2}{2}}(\nabla\cdot R_P) \in \mathcal{C}_0(\mathbb{R}^d)$ and that

$$\widehat{f_P} = \gamma_d\ \mathcal{F}((-\Delta)^{\frac{1}{2}}(-\Delta)^{\frac{d-2}{2}}(\nabla\cdot R_P))$$

holds in $\mathcal{S}(\mathbb{R}^d)'$, where γ_d is the same constant as in (A). We deduce that

$$f_P = \gamma_d\ (-\Delta)^{\frac{1}{2}}(-\Delta)^{\frac{d-2}{2}}(\nabla\cdot R_P) = \mathcal{L}_d(R_P)$$

in $\mathcal{S}(\mathbb{R}^d)'$. Since both sides of this last equality are continuous, equality also holds pointwise over $\mathbb{R}^d$, which concludes the proof. ■

Theorem 3.3. *Let $d \geq 1$ and P a Borel probability measure on $\mathbb{R}^d$. The equality $P = \mathcal{L}_d(R_P)$ holds in $\mathcal{S}(\mathbb{R}^d)'$, i.e. we have*

$$\forall\ \psi \in \mathcal{S}(\mathbb{R}^d), \qquad \int_{\mathbb{R}^d}\psi(x)\,dP(x) = \int_{\mathbb{R}^d}\Big(R_P(x), (\mathcal{L}_d^*\psi)(x)\Big)\,dx.$$

Proof of Theorem 3.3. Assume first that there exists a sequence of Borel probability measures (Q_k) on $\mathbb{R}^d$ such that (Q_k) converges in law to P as $k\to\infty$, and such that Q_k admits a density $f_k \in \mathcal{S}(\mathbb{R}^d)$ with respect to the Lebesgue measure for any k. Let R_{Q_k}

denote the geometric rank associated to the probability measure Q_k, for any k. Because Q_k admits the density $f_k \in \mathcal{S}(\mathbb{R}^d)$, Theorem 3.2 entails that

$$\forall\, k,\ \forall\, x \in \mathbb{R}^d, \qquad f_k(x) = (\mathcal{L}_d R_{Q_k})(x). \tag{3.5}$$

Fix $\psi \in \mathcal{S}(\mathbb{R}^d)$. For any k, (3.5) reads

$$\int_{\mathbb{R}^d} \psi(x) f_k(x)\, dx = \int_{\mathbb{R}^d} \Big(R_{Q_k}(x), (\mathcal{L}_d^* \psi)(x)\Big)\, dx. \tag{3.6}$$

We are going to show that the l.h.s. of (3.6) converges to $\int_{\mathbb{R}^d} \psi(x)\, dP(x)$, and the r.h.s. of (3.6) converges to

$$\int_{\mathbb{R}^d} \Big(R_P(x), (\mathcal{L}_d^* \psi)(x)\Big)\, dx$$

as $k \to \infty$. Because (Q_k) converges in law to P as $k \to \infty$, and ψ is continuous and bounded over $\mathbb{R}^d$, we have

$$\int_{\mathbb{R}^d} \psi(x) f_k(x)\, dx = \int_{\mathbb{R}^d} \psi(x)\, dQ_k(x) \to \int_{\mathbb{R}^d} \psi(x)\, dP(x) \tag{3.7}$$

as $k \to \infty$. Let us now show that the r.h.s. of (3.6) converges. We show first that R_{Q_k} converges almost everywhere to R_P as $k \to \infty$. For any $x \in \mathbb{R}^d$, define $g_x(z) := \frac{x-z}{\|x-z\|}\mathbb{I}[z \neq x]$ for any $z \in \mathbb{R}^d$. With the notations of Lemma A.4, we have $D_{g_x} = \{x\}$. Define $A := \{x \in \mathbb{R}^d : P[\{x\}] > 0\}$. Then, A is at most countable and we have $P[D_{g_x}] = 0$ for any $x \in \mathbb{R}^d \setminus A$. Because g_x is bounded and measurable for any $x \in \mathbb{R}^d$, Lemma A.4 entails that

$$\forall\, x \in \mathbb{R}^d \setminus A, \qquad R_{Q_k}(x) = \int_{\mathbb{R}^d} g_x(z)\, dQ_k(z) \to \int_{\mathbb{R}^d} g_x(z)\, dP(z) = R_P(x)$$

as $k \to \infty$. Because A is at most countable, R_{Q_k} converges to R_P almost everywhere as $k \to \infty$. To apply the dominated convergence theorem to the r.h.s. of (3.6), observe that $\mathcal{L}_d^*(\psi) \in L^1(\mathbb{R}^d)$. Indeed, if d is even we have $(-\Delta)^{\frac{d-2}{2}}\psi \in \mathcal{S}(\mathbb{R}^d)$ since $\psi \in \mathcal{S}(\mathbb{R}^d)$. It follows that

$$(-\Delta)^{\frac{1}{2}}((-\Delta)^{\frac{d-2}{2}}\psi) \in \mathcal{S}_{1/2}(\mathbb{R}^d),$$

whence

$$\mathcal{L}_d^*(\psi) = \gamma_d\, \nabla\Big((-\Delta)^{\frac{1}{2}}(-\Delta)^{\frac{d-2}{2}}\psi\Big) \in \mathcal{S}_{1/2}(\mathbb{R}^d, \mathbb{C}^d) \subset L^1(\mathbb{R}^d).$$

If d is odd, $\nabla\big((-\Delta)^{\frac{d-1}{2}}\psi\big)$ obviously belongs to $\mathcal{S}(\mathbb{R}^d, \mathbb{C}^d)$, which is a subset of $L^1(\mathbb{R}^d)$. Because the sequence of functions $(R_{Q_k})_k$ is uniformly norm-bounded by 1, and R_{Q_k} converges to R_P almost everywhere as $k \to \infty$, Lebesgue's dominated convergence theorem entails that

$$\int_{\mathbb{R}^d} \Big(R_{Q_k}(x), (\mathcal{L}_d^* \psi)(x)\Big)\, dx \to \int_{\mathbb{R}^d} \Big(R_P(x), (\mathcal{L}_d^* \psi)(x)\Big)\, dx \tag{3.8}$$

as $k \to \infty$. Putting (3.6), (3.7), and (3.8) together yields

$$\forall\, \psi \in \mathcal{S}(\mathbb{R}^d), \qquad \int_{\mathbb{R}^d} \psi(x)\, dP(x) = \int_{\mathbb{R}^d} (R_P, \mathcal{L}_d^*(\psi))\, dx.$$

It follows that $P = \mathcal{L}_d(R_P)$ in $\mathcal{S}(\mathbb{R}^d)'$.

It remains to show that there indeed exists a sequence of Borel probability measures (Q_k) on $\mathbb{R}^d$ converging in law to P as $k \to \infty$, such that Q_k admits a density $f_k \in \mathcal{S}(\mathbb{R}^d)$

with respect to the Lebesgue measure for any k. Let X be a random d-vector with law P. Let $\rho \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ be such that $0 \leq \rho \leq 1$ and $\int_{\mathbb{R}^d} \rho(x)\, dx = 1$. In particular, ρ is a probability density over $\mathbb{R}^d$. Then, let Y be a random d-vector with density ρ. For any k, define $X_k := X + \frac{1}{k}Y$. Because (X_k) converges to X in probability as $k \to \infty$, (X_k) converges in law to X as $k \to \infty$. Furthermore, observe that X_k admits the density $s_k := \rho_k * P$ with respect to the Lebesgue measure, where $\rho_k(x) := k^d \rho(kx)$ for any k and $x \in \mathbb{R}^d$. In particular, $s_k \in \mathcal{C}^\infty(\mathbb{R}^d)$ since $\rho_k \in \mathcal{C}_c^\infty(\mathbb{R}^d)$. Because (X_k) converges in law to X as $k \to \infty$, we have

$$\int_{\mathbb{R}^d} g(x) s_k(x)\, dx \to \int_{\mathbb{R}^d} g(x)\, dP(x)$$

for any continuous and bounded map $g : \mathbb{R}^d \to \mathbb{C}$ as $k \to \infty$. For any k, let $r_k > 0$ be such that

$$\int_{\mathbb{R}^d \setminus B_{r_k}} s_k(x)\, dx < \frac{1}{k}.$$

For any k, let $\chi_k \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ be such that $0 \leq \chi_k \leq 1$ over $\mathbb{R}^d$, $\chi_k = 1$ over B_{r_k}, and $\chi_k = 0$ over $\mathbb{R}^d \setminus B_{1+r_k}$. Then, define $f_k(x) := \chi_k(x) s_k(x)$ for any k and $x \in \mathbb{R}^d$. Because $(s_k) \subset \mathcal{C}^\infty(\mathbb{R}^d)$, we have $(f_k) \subset \mathcal{C}_c^\infty(\mathbb{R}^d)$. In particular, $(f_k) \subset \mathcal{S}(\mathbb{R}^d)$. Let $g : \mathbb{R}^d \to \mathbb{C}$ be a continuous and bounded map. We have

$$\Big| \int_{\mathbb{R}^d} g(x) f_k(x)\, dx - \int_{\mathbb{R}^d} g(x) s_k(x)\, dx \Big| \leq \int_{\mathbb{R}^d} |g(x)| (\chi_k(x) - 1) s_k(x)\, dx$$
$$\leq \int_{\mathbb{R}^d \setminus B_{r_k}} |g(x)| s_k(x)\, dx \leq \|g\|_{L^\infty(\mathbb{R}^d)} \int_{\mathbb{R}^d \setminus B_{r_k}} s_k(x)\, dx \leq \frac{1}{k} \|g\|_{L^\infty(\mathbb{R}^d)}.$$

Because $\int_{\mathbb{R}^d} g(x) s_k(x)\, dx \to \int_{\mathbb{R}^d} g(x)\, dP(x)$ as $k \to \infty$, we deduce that

$$\int_{\mathbb{R}^d} g(x) f_k(x)\, dx \to \int_{\mathbb{R}^d} g(x)\, dP(x)$$

for any continuous and bounded map $g : \mathbb{R}^d \to \mathbb{C}$ as $k \to \infty$. Letting Q_k be the probability measure with density $f_k \in \mathcal{S}(\mathbb{R}^d)$ for any k yields the conclusion. ∎

3.2. **Pointwise recovery.** In Section 3.1, we established that any probability measure P on $\mathbb{R}^d$ writes $P = \mathcal{L}_d(R_P)$ in the sense of tempered distributions, where

$$\mathcal{L}_d = \gamma_d \, (-\Delta)^{\frac{d-1}{2}} \nabla \cdot$$

is the operator introduced at the beggining of Section 3. We further showed that if P admits a smooth and fast-decreasing density $f_P \in \mathcal{S}(\mathbb{R}^d)$, the equality $f_P = \mathcal{L}_d(R_P)$ actually holds pointwise over $\mathbb{R}^d$. In this section, we give less restrictive conditions on f_P to ensure that $f_P = \mathcal{L}_d(R_P)$ still holds pointwise. In this case, one can compute $(\mathcal{L}_d R_P)(x)$ by successively applying the differential operators involved in the definition of $\mathcal{L}_d$ to R_P pointwise at x. In the univariate case, minimal assumptions are already well-known. Indeed, recall that $\mathcal{L}_1 = \frac{1}{2}\frac{d}{dx}$ and $R_P = (2F_P - 1)$ when $d = 1$, where $F_P(x) = \int_{-\infty}^x f_P(s)\, ds$ is the cdf of P. Provided f_P is continuous in a neighbourhood of $x_0 \in \mathbb{R}$, the fundamental theorem of calculus yields

$$f_P(x_0) = \frac{dF_P}{dx}(x_0) = (\mathcal{L}_1 R_P)(x_0).$$

In dimension $d \geq 2$, the situation is different. Indeed, computing the derivatives of $R_P = K_d * f_P$ requires differentiating K_d. Unlike in the one-dimensional case, where K_1 is just the sign function, the derivatives $\partial^\alpha K_d$ of K_d behave like $1/|x|^{|\alpha|}$ and display a singularity at the origin. This makes the identification of differentiability properties of R_P more difficult. In particular, we will not be able to conclude that R_P is of class $\mathcal{C}^d$ by requiring only $f_P \in \mathcal{C}^0$. We will, however, prove that this is the case under the additional assumption $f_P \in \mathcal{C}^{0,\beta}$ for some $\beta \in (0, 1)$. Let us stress the fact that the values $\beta \in \{0, 1\}$ are discarded. In particular, our results do not apply to continuous functions without further requirements. This is a consequence of the fact that continuity of the weak Laplacian Δu of some appropriate u does not yield twice differentiability of u in general; see the beginning of Section 2.2 in [24] for further details on this question. Indeed this last property, which is the content of Proposition 2.7 and Corollary 2.8, is the key ingredient we use to establish the differentiability of R_P up to order d in Theorem 3.5.

Computing $\mathcal{L}_d(R_P)$ pointwise requires R_P to be at least $d-1$ times differentiable : d deratives are needed when d is odd, $d-1$ derivatives and one pseudo-derivative when d is even. In each of these cases, R_P should be of class $\mathcal{C}^{d-1}$. We prove that R_P reaches this regularity under very weak assumptions in Proposition 3.4. They consist of *a priori* regularity statements of R_P up to order $d-1$. Because the dth derivatives of R_P are strongly related to deconvolution of $R_P = K * f_P$ (at least formally, we have $\mathcal{L}_d K = \delta$, the Dirac distribution), reaching differentiability of order d is more challenging and calls upon different strategies. Once we establish that $f_P = \mathcal{L}_d R_P$ holds pointwise, we will be able to improve on the *a priori* regularity of R_P using elliptic regularity.

As a preliminary result, let us mention that a straightforward application of Lebesgue's dominated convergence theorem yields that R_P is continuous at a point $x \in \mathbb{R}^d$ if and only if $P[\{x\}] = 0$. In particular, R_P is continuous over $\mathbb{R}^d$ when P admits a density with respect to the Lebesgue measure.

Proposition 3.4. *Fix $d \geq 2$, $\Omega \subset \mathbb{R}^d$ an open subset, and an integer $\ell \in [1, d-1]$. Let P be a Borel probability measure on $\mathbb{R}^d$, and assume that P admits a density $f_\Omega \in L^1(\Omega)$ over Ω with respect to the Lebesgue measure.*

(1) If $f_\Omega \in L^p_{loc}(\Omega)$ for some $p \in (\frac{d}{d-\ell}, \infty]$, then $R_P \in \mathcal{C}^\ell(\Omega)$ and $\partial^\alpha R_P(x) = \mathbb{E}[(\partial^\alpha K)(x - Z)]$ for any $x \in \Omega$ and $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq \ell$.

(2) If $\Omega = \mathbb{R}^d$ and the density belongs to $L^p(\mathbb{R}^d)$ for some $p \in (\frac{d}{d-\ell}, \infty]$, then $\partial^\alpha R_P$ converges to 0 at infinity for any $\alpha \in \mathbb{N}^d$ such that $1 \leq |\alpha| \leq \ell$.

The requirement that the density f_Ω be in L^p_{loc} for some $p > \frac{d}{d-\ell}$ might feel unnatural in the first statement of Proposition 3.4. If f_Ω is bounded in a neighbourhood of x, the result is a straightforward application of Lebesgue's dominated convergence theorem. Our integrability condition, however, allows singularities for f_Ω, but prevents them from growing too fast. For instance, if $f_\Omega(z)$ behaves like $|z|^{-\beta}$ around $z = 0$ for some $\beta > 0$, then f_Ω will be integrable to the pth power around 0 provided $\beta < d/p$. Because p can be taken arbitrarily close to $d/(d-\ell)$, this is equivalent to $\beta < d-\ell$ without loss of generality. The latter condition has a clear interpretation: the more derivatives we want, i.e. the higher we take ℓ, the more derivatives of K must be integrable, the smaller the spikes of f_Ω should be.

Because the operator $\mathcal{L}_d$ varies with d, we split the next result in two distinct theorems. We start by establishing the pointwise equality $f_P = \mathcal{L}_d(R_P)$ when d is odd.

Theorem 3.5 (Odd dimension)**.** *Let $d \geq 3$ be odd, $\Omega \subset \mathbb{R}^d$ an open subset, and P a Borel probability measure on $\mathbb{R}^d$ admititing a density $f_\Omega \in L^1(\Omega)$ over Ω with respect to the Lebesgue measure. Fix $\beta \in (0,1)$ and $k \in \mathbb{N}$. If $f_\Omega \in \mathcal{C}^{k,\beta}_{loc}(\Omega)$, then $R_P \in \mathcal{C}^{d+k,\beta}_{loc}(\Omega)$ and $f_\Omega(x) = (\mathcal{L}_d R_P)(x)$ for any $x \in \Omega$.*

Proof of Theorem 3.5. Because P is non-atomic over Ω, recall that $\nabla g_P = R_P$ over Ω (see Definition 1.1 and the comments below). Since $\nabla \cdot R_P = \Delta g_P$, Theorem 3.3 entails that $f_\Omega = -\gamma_d\,(-\Delta)^{\frac{d+1}{2}} g_P$ in $\mathcal{S}(\mathbb{R}^d)'$. The fact that $f_\Omega \in \mathcal{C}^{k,\beta}_{\text{loc}}(\Omega)$ entails that $f_\Omega \in L^p_{\text{loc}}(\Omega)$ for any $p > n$. Consequently, Proposition 3.4 yields $R_P \in \mathcal{C}^{d-1}(\Omega)$, hence $g_P \in \mathcal{C}^d(\Omega)$.

Define $h_0 = -f_\Omega$ and $h_j := \gamma_d\,(-\Delta)^{\frac{d+1}{2}-j} g_P$ for all $j \in \{1,\dots,(d+1)/2\}$. Let us show recursively that $h_j \in \mathcal{C}^{k+2j,\beta}_{\text{loc}}(\Omega)$ for all $j \in \{0,\dots,(d+1)/2\}$. For $j = 0$, this follows from the fact that f_Ω belongs to $\mathcal{C}^{k,\beta}_{\text{loc}}(\Omega)$. Fix $j \in \{0,\dots,(d-1)/2\}$ and assume that $h_j \in \mathcal{C}^{k+2j,\beta}_{\text{loc}}(\Omega)$, by induction. We need to show that h_{j+1} belongs to $\mathcal{C}^{k+2j+2,\beta}_{\text{loc}}(\Omega)$. To that end, observe that h_{j+1} belongs to $H^1_{\text{loc}}(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$. Indeed, the fact that $g_P \in \mathcal{C}^d(\Omega)$ entails that

$$h_{j+1} \in \mathcal{C}^{2j+1}(\Omega) \subset \mathcal{C}^1(\Omega) \subset H^1_{\text{loc}}(\Omega) \cap L^\infty_{\text{loc}}(\Omega).$$

Because $-\Delta h_{j+1} = h_j$ in $\mathcal{S}(\mathbb{R}^d)'$ with $h_j \in \mathcal{C}^{k+2j,\beta}_{\text{loc}}(\Omega)$ and $h_{j+1} \in H^1_{\text{loc}}(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$, Corollary 2.8 yields that $h_j \in \mathcal{C}^{k+2j+2,\beta}_{\text{loc}}(\Omega)$. It follows by induction that $g_P = h_{(d+1)/2}$ belongs to $\mathcal{C}^{d+k+1,\beta}_{\text{loc}}(\Omega)$. Consequently, $R_P = \nabla g_P$ belongs to $\mathcal{C}^{d+k,\beta}_{\text{loc}}(\Omega)$. Because f_Ω is continuous over Ω and R_P is of class $\mathcal{C}^d$ on Ω, the fact that $f_\Omega = \mathcal{L}_d(R_P)$ in $\mathcal{S}(\mathbb{R}^d)'$ entails that equality actually holds pointwise over Ω. ■

Because $\mathcal{L}_d$ is a purely local operator - it involes only an integer number of times the usual Laplacian $-\Delta$ - when d is odd, Theorem 3.5 can be localised over an open set $\Omega \subset \mathbb{R}^d$. However, this is not the case anymore when d is even, as can be seen from (2.1): the value of $(-\Delta)^{1/2}u$ at a point $x \in \mathbb{R}^d$ depends on the values of u over all of $\mathbb{R}^d$. Consequently, Theorem 3.6, the analogue of Theorem 3.5 when d is even, is not a local result. We discuss this question in more details in Section 6.

Theorem 3.6 (Even dimension)**.** *Let $d \geq 2$ be even and P a Borel probability measure on $\mathbb{R}^d$ admitting a density $f_P \in L^1(\mathbb{R}^d)$ with respect to the Lebesgue measure. Fix $\beta \in (0,1)$ and $k \in \mathbb{N}$. If f_P belongs to $\mathcal{C}^{k,\beta}(\mathbb{R}^d)$, then $R_P \in \mathcal{C}^{d+k,\beta}_{loc}(\mathbb{R}^d)$ and $f_P(x) = (\mathcal{L}_d R_P)(x)$ for any $x \in \mathbb{R}^d$. In addition, letting*

$$R_P^{(d-1)} := \gamma_d\,(-\Delta)^{\frac{d-2}{2}}(\nabla \cdot R_P),$$

we have

$$\forall\, x \in \mathbb{R}^d, \qquad (\mathcal{L}_d R_P)(x) = c_{d,1/2} \lim_{\eta \to 0} \int_{\mathbb{R}^d \setminus B_\eta(x)} \frac{R_P^{(d-1)}(x) - R_P^{(d-1)}(z)}{|x-z|^{d+1}}\, dz;$$

see (2.2) in Section 2.3 for the definition of $c_{d,1/2}$.

Proof of Theorem 3.6. Firstly, let us show that $R_P^{(d-1)} \in \mathcal{C}^{1,\beta}(\mathbb{R}^d)$. Because $f_P \in \mathcal{C}^{0,\beta}(\mathbb{R}^d)$, we have $f_P \in L^\infty(\mathbb{R}^d)$. Therefore, Theorem 3.4 yields that R_P belongs to

$\mathcal{C}_b^{d-1}(\mathbb{R}^d)$. In particular, $R_P^{(d-1)}$ is bounded over $\mathbb{R}^d$. Because $f_P = (-\Delta)^{1/2} R_P^{(d-1)}$ in $\mathcal{S}(\mathbb{R}^d)'$ (see Theorem 3.3) with $R_P^{(d-1)} \in L^\infty(\mathbb{R}^d)$ and $f_P \in \mathcal{C}^{0,\beta}(\mathbb{R}^d)$, Proposition 2.8 in [27] entails that $R_P^{(d-1)} \in \mathcal{C}^{1,\beta}(\mathbb{R}^d)$. In addition, Proposition 2.12 yields

$$(\mathcal{L}_d R_P)(x) = (-\Delta)^{1/2} R_P^{(d-1)}(x) = c_{d,1/2} \lim_{\eta \to 0} \int_{\mathbb{R}^d \setminus B_\eta(x)} \frac{R_P^{(d-1)}(x) - R_P^{(d-1)}(z)}{|x-z|^{d+1}}\, dz$$

for any $x \in \mathbb{R}^d$.

Secondly, we prove that $R_P^{(d-1)} \in \mathcal{C}_{\text{loc}}^{k+1,\beta}(\mathbb{R}^d)$. Because we just proved a stronger version when $k = 0$, assume that $k \geq 1$. Since $f_P \in L^1(\mathbb{R}^d)$ and $L^1(\mathbb{R}^d) \subset \mathcal{S}_{1/2}(\mathbb{R}^d)'$, then $(-\Delta)^{1/2} f_P$ is well-defined. Because $f_P \in \mathcal{C}^{k,\beta}(\mathbb{R}^d)$, Proposition 2.7 in [27] entails that $(-\Delta)^{1/2} f_P \in \mathcal{C}^{k-1,\beta}(\mathbb{R}^d)$. It follows that $(-\Delta)^{1/2} f_P = -\Delta R_P^{(d-1)}$ in $\mathcal{S}(\mathbb{R}^d)'$, with $(-\Delta)^{1/2} f_P \in \mathcal{C}^{k-1,\beta}(\mathbb{R}^d)$ and $R_P^{(d-1)} \in H_{\text{loc}}^1(\mathbb{R}^d) \cap L_{\text{loc}}^\infty(\mathbb{R}^d)$ (recall that $R_P^{(d-1)} \in \mathcal{C}^{1,\beta}(\mathbb{R}^d)$). Consequently, Corollary 2.8 yields that $R_P^{(d-1)} \in \mathcal{C}_{\text{loc}}^{k+1,\beta}(\mathbb{R}^d)$.

Finally, we show that $R_P \in \mathcal{C}_{\text{loc}}^{d+k,\beta}(\mathbb{R}^d)$. Because P is non-atomic over Ω, recall that $\nabla g_P = R_P$ over $\mathbb{R}^d$ (see Definition 1.1 and the comments below). Because $\nabla \cdot R_P = \Delta g_P$, we have $-R_P^{(d-1)} = \gamma_d\, (-\Delta)^{d/2} g_P$ in $\mathcal{S}(\mathbb{R}^d)'$. Furthermore, $(-\Delta)^{\frac{d}{2}-k} g_P$ belongs to $H_{\text{loc}}^1(\mathbb{R}^d) \cap L_{\text{loc}}^\infty(\mathbb{R}^d)$ for any $k \in \{1, \dots, \frac{d}{2}\}$ since g_P belongs to $\mathcal{C}_b^d(\mathbb{R}^d)$. Therefore, the same argument as in the proof of Theorem 3.5 yields that g_P belongs to $\mathcal{C}_{\text{loc}}^{d+k+1,\beta}(\mathbb{R}^d)$, since $R_P^{(d-1)} \in \mathcal{C}_{\text{loc}}^{k+1,\beta}(\mathbb{R}^d)$. In particular, we have $R_P \in \mathcal{C}_{\text{loc}}^{d+k,\beta}(\mathbb{R}^d)$. ■

4. Examples

In this section we compute $\gamma_d\ \mathcal{L}_d(R_P)$ when P is a standard normal and a standard Cauchy distribution in $\mathbb{R}^2$ and $\mathbb{R}^3$. We show that the result coincides with the density of P, as established in Section 3. Because the nature of the operator $\mathcal{L}_d$ depends on whether d is odd or even, computing $\mathcal{L}_2$ and $\mathcal{L}_3$ is fundamentally different.

Recall that the standard normal and standard Cauchy distributions on $\mathbb{R}^d$ have density

$$f(x) = \frac{1}{(2\pi)^{d/2}} \exp(-|x|^2/2), \quad \text{and} \quad f(x) = \frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}} \frac{1}{(1+|x|^2)^{\frac{d+1}{2}}},$$

respectively. Because these distributions are spherically symmetric, Proposition 2.2 (i) in [18] entails that there exists a function $g : [0,\infty) \to [0,\infty)$ such that their geometric rank R_P writes

$$\forall\, x \in \mathbb{R}^d, \qquad R_P(x) = g(|x|)\frac{x}{|x|}.$$

When P is the standard normal distribution on $\mathbb{R}^d$, it was shown in Section 4.3 of [13] that

$$\forall\, r > 0, \qquad g(r) = \frac{1}{\sqrt{2}} \frac{\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d+2}{2})}\, r \exp(-r^2/2)\ {}_1F_1\Big(\frac{d+1}{2}; \frac{d+2}{2}; \frac{r^2}{2}\Big), \tag{4.1}$$

where ${}_1F_1$ is the confluent hypergeometric function; see, e.g., (13.2.2) in [29]. When P is the standard Cauchy distribution on $\mathbb{R}^d$, it was shown in Section 4.3 of [13] that

$$\forall\, r>0, \qquad g(r)=\frac{1}{\sqrt{\pi}}\frac{\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d+2}{2})}\frac{r}{1+r^2}\,{}_2F_1\Big(\frac{d+1}{2},1;\frac{d+2}{2};\frac{r^2}{1+r^2}\Big), \tag{4.2}$$

where ${}_2F_1$ is the Gaussian hypergeometric function; see, e.g., (15.2.1) in [29].

Both when $d=2$ and $d=3$, we need to compute the divergence $\nabla\cdot R_P=\sum_{i=1}^d \partial_i (R_P)_i$ of R_P. A straightforward computation gives

$$\forall\, x\in\mathbb{R}^d, \qquad (\nabla\cdot R_P)(x)=g'(|x|)+(d-1)\frac{g(|x|)}{|x|}. \tag{4.3}$$

4.1. **Dimension 3.** Letting $h(|x|):=(\nabla\cdot R_P)(x)$ for any $x\in\mathbb{R}^3$, (4.3) yields $h(r)=g'(r)+2g(r)/r$ for any $r>0$. Recalling that $\gamma_3=(8\pi)^{-1}$ (see the beginning of Section 3), a straightforward computation yields

$$\forall\, x\in\mathbb{R}^3, \qquad (\mathcal{L}_3 R_P)(x)=\gamma_3\,(-\Delta)\big(\nabla\cdot R_P\big)(x)=-\frac{1}{8\pi}\Big(h''(|x|)+2\frac{h'(|x|)}{|x|}\Big).$$

Letting $e\in{}^2$ be arbitrary, we then need to show that

$$\forall\, r>0, \qquad -\frac{1}{8\pi}\Big(h''(r)+2\frac{h'(r)}{r}\Big)=f(re),$$

where $h(r)=g'(r)+2g(r)/r$ for any $r>0$.

4.1.1. *Trivariate Gaussian distribution.* Taking $d=3$ in (4.1) yields

$$\forall\, r>0, \qquad g(r)=\frac{2\sqrt{2}}{3\sqrt{\pi}}\,r\exp(-r^2/2)\,{}_1F_1\Big(2;\frac{5}{2};\frac{r^2}{2}\Big)=\frac{2\sqrt{2}}{3\sqrt{\pi}}\,{}_1F_1\Big(\frac{1}{2};\frac{5}{2};-\frac{r^2}{2}\Big),$$

where the last equality follows from (13.2.39) in [29]. Using (13.3.2) in [29] with $a=\frac{1}{2}$ and $b=\frac{3}{2}$ gives

$$\begin{aligned}\forall\, r>0, \qquad g(r) &= \frac{4\sqrt{2}}{3\sqrt{\pi}r}\Big\{\frac{r^2}{2}\,{}_1F_1\Big(\frac{1}{2};\frac{5}{2};-\frac{r^2}{2}\Big)\Big\}\\ &= \frac{4\sqrt{2}}{3\sqrt{\pi}r}\Big\{\frac{3}{4}\,{}_1F_1\Big(\frac{1}{2};\frac{1}{2};-\frac{r^2}{2}\Big)+\frac{3}{2}\Big(\frac{r^2}{2}-\frac{1}{2}\Big)\,{}_1F_1\Big(\frac{1}{2};\frac{3}{2};-\frac{r^2}{2}\Big)\Big\}\\ &= \frac{4\sqrt{2}}{3\sqrt{\pi}r}\Big\{\frac{3}{4}\exp(-r^2/2)+\frac{3}{4}(r^2-1)\,\frac{\sqrt{\pi}}{\sqrt{2}r}\,\mathrm{erf}(r/\sqrt{2})\Big\},\end{aligned}$$

where the last equality follows from (13.6.1) and (13.6.7) in [29]; here,

$$\mathrm{erf}(r)=\frac{2}{\sqrt{\pi}}\int_0^r \exp(-t^2)\,dt=2\Phi(\sqrt{2}r)-1, \qquad r>0$$

is the error function (throughout, Φ and ϕ stand for the cumulative distribution function and probability density function of the standard normal distribution, respectively). Thus,

$$\forall\, r>0, \qquad g(r) = \frac{\sqrt{2}}{\sqrt{\pi}r}\exp(-r^2/2) + \frac{r^2-1}{r^2}\,\mathrm{erf}(r/\sqrt{2}) = \frac{2}{r}\phi(r) + \frac{r^2-1}{r^2}(2\Phi(r)-1)\cdot$$

Using the fact that $\phi'(r) = -r\phi(r)$ for any $r>0$, straightforward computations give

$$\forall\, r>0, \qquad g'(r) = 2\,\frac{2\Phi(r)-1-2r\phi(r)}{r^3}.$$

This yields

$$\forall\, r>0, \qquad h(r) = g'(r) + 2\frac{g(r)}{r} = \frac{2(2\Phi(r)-1)}{r}.$$

It follows that $h'(r) = \frac{4r\phi(r)-4\Phi(r)+2}{r^2}$, and $h''(r) = -4\phi(r) - 2\frac{4r\phi(r)-4\Phi(r)+2}{r^3}$ for any $r>0$. Hence,

$$\forall\, r>0, \qquad -\frac{1}{8\pi}\left(h''(r) + \frac{2}{r}h'(r)\right) = \frac{\phi(r)}{2\pi} = \frac{1}{(2\pi)^{3/2}}\exp(-r^2/2) = f(re),$$

which, as expected, coincides with the probability density function of the trivariate standard normal distribution.

4.1.2. *Trivariate Cauchy distribution.* Taking $d=3$ in (4.2) yields

$$\forall\, r>0, \qquad g(r) = \frac{4r}{3\pi(1+r^2)}\,{}_2F_1\Big(2,1;\frac{5}{2};\frac{r^2}{1+r^2}\Big) = \frac{4r}{3\pi\sqrt{1+r^2}}\,{}_2F_1\Big(\frac{1}{2},\frac{3}{2};\frac{5}{2};\frac{r^2}{1+r^2}\Big),$$

where we used (15.8.1) in [29]. Applying Identity 92 on page 473 of [30] then provides

$$\forall\, r>0, \qquad g(r) = \frac{2}{\pi r^2}\Big((1+r^2)\arcsin\Big(\frac{r}{\sqrt{1+r^2}}\Big) - r\Big) = \frac{2}{\pi r^2}\Big((1+r^2)\arctan(r) - r\Big).$$

Direct computations then yield

$$\forall\, r>0, \qquad h(r) = g'(r) + 2\frac{g(r)}{r} = \frac{4\arctan(r)}{\pi r},$$

hence

$$\forall\, r>0, \qquad -\frac{1}{8\pi}\left(h''(r) + \frac{2}{r}h'(r)\right) = \frac{1}{\pi^2(1+r^2)} = f(re),$$

which coincides with the probability density function of the trivariate standard Cauchy distribution.

4.2. **Dimension 2.** Recall that $(-\Delta)^{1/2}u$ is defined through $((-\Delta)^{1/2}u)(x) = 2\pi\,\mathcal{F}^{-1}(|\xi|\mathcal{F}u(\xi))(x)$ for any $x\in\mathbb{R}^2$ (see Section 2.3). Because $\Gamma(3/2)=\sqrt{\pi}/2$, we have $\gamma_2=(2\pi)^{-1}$ (see the beginning of Section 3). Letting $u=\nabla\cdot R_P$, it follows that

$$\forall\, x\in\mathbb{R}^2, \qquad (\mathcal{L}_2R_P)(x) = \gamma_2\,(-\Delta)^{1/2}(\nabla\cdot R_P)(x) = \mathcal{F}^{-1}\big(|\xi|\mathcal{F}u(\xi)\big)(x).$$

Writing $u(x)=h(|x|)$ for any $x\in\mathbb{R}^2$, (4.3) yields $h(r)=g'(r)+g(r)/r$ for any $r>0$. A straightforward computation gives

$$\forall\, \xi\in\mathbb{R}^2, \qquad (\mathcal{F}u)(\xi) = \int_0^\infty h(r)\left(\int_0^{2\pi} e^{-i(2\pi r|\xi|)\cos\theta}\,d\theta\right) r\,dr = 2\pi\int_0^\infty h(r)J_0(2\pi r|\xi|)r\,dr,$$

where $J_0(z) = (2\pi)^{-1}\int_0^{2\pi} e^{-iz\cos\theta}\,d\theta$ is the Bessel function of the first kind with order zero. Writing $\tilde{h}$ for the function defined through $\tilde{h}(r) = \sqrt{r}h(r)$, we thus have

$$(\mathcal{F}u)(\xi) = \sqrt{\frac{2\pi}{|\xi|}}\int_0^\infty \sqrt{r}h(r)J_0(r(2\pi|\xi|))\sqrt{r(2\pi|\xi|)}\,dr = \sqrt{\frac{2\pi}{|\xi|}}(\mathcal{H}_0\tilde{h})(2\pi|\xi|),$$

where $(\mathcal{H}_0\phi)(r) := \int_0^\infty \phi(s)J_0(sr)\sqrt{sr}\,ds$ is the Hankel transform of ϕ with order zero; see, e.g., page 1 in [31]. Finally, to compute $\mathcal{F}^{-1}(|\xi|\mathcal{F}u(\xi))(x)$ we will use the fact that the restriction of $\mathcal{F}$ and $\mathcal{F}^{-1}$ to istropic functions coincide.

4.2.1. *Bivariate Gaussian distribution.* Taking $d = 2$ in (4.1) yields

$$\forall\, r > 0, \qquad g(r) = \frac{\sqrt{\pi}}{2\sqrt{2}}\, r\exp(-r^2/2)\; {}_1F_1\Big(\frac{3}{2}; 2; \frac{r^2}{2}\Big) = \frac{\sqrt{\pi}}{\sqrt{2r}}\Big\{\frac{r^2}{2}\exp(-r^2/2)\; {}_1F_1\Big(\frac{3}{2}; 2; \frac{r^2}{2}\Big)\Big\}.$$

Hence, applying (13.3.21) in [29] provides

$$\begin{aligned}\forall\, r > 0, \qquad g'(r) &= -\frac{\sqrt{\pi}}{2\sqrt{2}}\exp(-r^2/2)\; {}_1F_1\Big(\frac{3}{2}; 2; \frac{r^2}{2}\Big) + \frac{\sqrt{\pi}}{\sqrt{2r}}\Big\{\exp(-r^2/2)\; {}_1F_1\Big(\frac{1}{2}; 1; \frac{r^2}{2}\Big)\Big\}r \\ &= -\frac{g(r)}{r} + \frac{\sqrt{\pi}}{\sqrt{2}}\exp(-r^2/2)\; {}_1F_1\Big(\frac{1}{2}; 1; \frac{r^2}{2}\Big).\end{aligned}$$

Therefore, (13.6.9) in [29] yields

$$\forall\, r > 0, \qquad h(r) = g'(r) + \frac{g(r)}{r} = \frac{\sqrt{\pi}}{\sqrt{2}}\exp(-r^2/2)\; {}_1F_1\Big(\frac{1}{2}; 1; \frac{r^2}{2}\Big) = \frac{\sqrt{\pi}}{\sqrt{2}}\exp(-r^2/4)\mathcal{I}_0\Big(\frac{r^2}{4}\Big),$$

where $\mathcal{I}_0$ is the modified Bessel function of order 0. Using (2.126) in [31] with $a = 1/4$, we obtain that $u(x) = h(|x|)$ satisfies

$$(\mathcal{F}u)(\xi) = \sqrt{\frac{2\pi}{|\xi|}}(\mathcal{H}_0\tilde{h})(2\pi|\xi|) = \sqrt{\frac{2\pi}{|\xi|}}\frac{\sqrt{\pi}}{\sqrt{2}}\frac{1}{\sqrt{(\pi/2)2\pi|\xi|}}\exp(-(2\pi|\xi|)^2/2) = \frac{1}{|\xi|}\exp(-2\pi^2|\xi|^2),$$

so that $|\xi|(\mathcal{F}u)(\xi) = \exp(-2\pi^2|\xi|^2)$. Using (2.23) in [31] with $a = 2\pi^2$, we then have

$$(\mathcal{F}^{-1}(|\xi|\mathcal{F}u(\xi)))(x) = \sqrt{\frac{2\pi}{|x|}}\Big(\frac{1}{4\pi^2}\sqrt{2\pi|x|}\exp(-4\pi^2|x|^2/(8\pi^2))\Big) = \frac{1}{2\pi}\exp(-|x|^2/2).$$

Thus,

$$\forall\, x \in \mathbb{R}^2, \qquad (\mathcal{L}_2R_P)(x) = \mathcal{F}^{-1}(|\xi|\mathcal{F}u(\xi))(x) = \frac{1}{2\pi}\exp(-|x|^2/2),$$

which is indeed the probability density function of the bivariate standard normal distribution.

4.2.2. *Bivariate Cauchy distribution.* Taking $d = 2$ in (4.2) yields

$$\forall\, r > 0, \qquad g(r) = \frac{r}{2(1+r^2)}\; {}_2F_1\Big(\frac{3}{2}, 1; 2; \frac{r^2}{1+r^2}\Big).$$

Hence, applying (15.8.1) in [29], then Identity 84 on page 473 of [30], provides

$$\forall\, r > 0, \qquad g(r) = \frac{r}{2\sqrt{1+r^2}}\; {}_2F_1\Big(\frac{1}{2}, 1; 2; \frac{r^2}{1+r^2}\Big) = \frac{r}{1+\sqrt{1+r^2}}.$$

Therefore, direct computation yields

$$h(r) = g'(r) + \frac{g(r)}{r} = \frac{1}{\sqrt{1+r^2}}\cdot$$

Now, using (2.19) in [31] with $a = 1$, we obtain that $u(x) = h(|x|)$ satisfies

$$\forall\ \xi \in \mathbb{R}^2, \qquad (\mathcal{F}u)(\xi) = \sqrt{\frac{2\pi}{|\xi|}}(\mathcal{H}_0\tilde{h})(2\pi|\xi|) = \sqrt{\frac{2\pi}{|\xi|}}\frac{1}{\sqrt{2\pi|\xi|}}\exp(-2\pi|\xi|),$$

so that $|\xi|\mathcal{F}u(\xi) = \exp(-2\pi|\xi|)$. Thus, using (2.23) in [31] with $a = 2\pi$, we have

$$(\mathcal{F}^{-1}(|\xi|\mathcal{F}u(\xi)))(x) = \sqrt{\frac{2\pi}{|x|}}\left(\frac{2\pi\sqrt{2\pi|x|}}{(4\pi^2 + (2\pi|x|)^2)^{3/2}}\right) = \frac{1}{2\pi(1+|x|^2)^{3/2}}$$

It follows that

$$\forall\ x \in \mathbb{R}^2, \qquad (\mathcal{L}_2 R_P)(x) = \mathcal{F}^{-1}(|\xi|\mathcal{F}u(\xi))(x) = \frac{1}{2\pi(1+|x|^2)^{3/2}},$$

which is the probability density function of the bivariate standard Cauchy distribution.

5. Depth regions

In this section, we prove new results about geometric depths regions of an arbitrary probability measure on $\mathbb{R}^d$. Firstly, we use the results we established in Section 3 to characterise the regularity of the depths regions. Secondly, letting Z be a random variable and F_Z its cdf, we provide an analogue of the formula $\mathbb{P}[a < Z \leq b] = F_Z(b) - F_Z(a)$ in the multivariate setting in terms of the geometric rank.

Consider a probability measure P on $\mathbb{R}^d$, with $d \geq 2$. For any $\beta \in [0,1)$ and $u \in {}^{d-1}$, recall that a geometric quantile of order β in direction u for P is an arbitrary minimizer of the objective function $O^P_{\beta,u}$, introduced in Section 1. When P is not supported on a single line of $\mathbb{R}^d$, Theorem 1 in [16] implies that the geometric quantile of order β in direction u for P is unique for any $\beta \in [0,1)$ and $u \in {}^{d-1}$; we denote it by $Q_P(\beta u)$. Under these assumptions, we define the geometric quantile regions $\mathcal{D}^\beta_P$ and contours $\mathcal{C}^\beta_P$ of arbitrary order $\beta \in [0,1)$ in the next definition.

Definition 5.1 (Depth contours and regions)**.** *Let $d \geq 2$ and P a probability measure on $\mathbb{R}^d$. Assume that P is not supported on a single line of $\mathbb{R}^d$. For any $\beta \in [0,1)$, we define the depth region $\mathcal{D}^\beta_P$ and depth contour $\mathcal{C}^\beta_P$ of order β for P by letting*

$$\mathcal{D}^\beta_P = \Big\{Q_P(\alpha u) : \alpha \in [0,\beta],\ u \in {}^{d-1}\Big\} \quad \textit{and} \quad \mathcal{C}^\beta_P = \Big\{Q_P(\beta u) : u \in {}^{d-1}\Big\}.$$

5.1. **Regularity.** When P is non-atomic and not supported on a line of $\mathbb{R}^d$, Proposition 6.1 in [14] entails that Q_P is a continuous map over the open unit ball B_1. It directly follows that $\mathcal{D}^\beta_P = Q_P(\beta\ \overline{B_1})$ is compact and arc-connected, and that $\mathcal{C}^\beta_P = Q_P(\beta\ {}^{d-1})$ is compact and arc-connected as well. Furthermore, the depth regions $(D^\beta_P)_{\beta\in[0,1)}$ are obivously nested, while depth contours $(\mathcal{C}^\beta_P)_{\beta\in[0,1)}$ are disjoint. Although depth regions are convex in most cases, they may fail to be convex in general ; see [32] for a detailed and quantified discussion of the shape of depth regions.

To state regularity properties of depth contours, let us first rewrite depth contours in terms of the rank map R_P. Theorem 6.1 in [14] entails that $x = Q_P(\alpha u)$ if and only if $R_P(x) = \alpha u$. This allows one to rewrite

$$\mathcal{D}_P^\beta = \Big\{x \in \mathbb{R}^d : |R_P(x)| \leq \beta\Big\} \quad \text{and} \quad \mathcal{C}_P^\beta = \Big\{x \in \mathbb{R}^d : |R_P(x)| = \beta\Big\}.$$

The results of Section 3 may now easily be used to derive regularity properties of depth contours, as we show in the next proposition.

Proposition 5.2. *Let $d \geq 2$ and P a probability measure on $\mathbb{R}^d$. Assume that P admits a density $f_P \in L^1(\mathbb{R}^d)$ with respect to the Lebesgue measure. Fix $\beta \in [0, 1)$.*

(1) If $f_P \in L^p_{loc}(\mathbb{R}^d)$ for some $p \in (\frac{d}{d-\ell}, \infty]$ and integer $\ell \in [1, d-1]$, then the depth contour $\mathcal{C}_P^\beta$ is a $(d-1)$-dimensional manifold of class $\mathcal{C}^\ell$;

(2) If $f_P \in \mathcal{C}^{k,\alpha}(\mathbb{R}^d)$ for some $k \in \mathbb{N}$ and $\alpha \in (0, 1)$, then the depth contour $\mathcal{C}_P^\beta$ is a $(d-1)$-dimensional manifold of class $\mathcal{C}^{k+d}$.

Proof of Proposition 5.2. Proposition 3.4, Theorem 3.5 and Theorem 3.6 yield that R_P has the stated regularity, $R_P \in \mathcal{C}^j(\mathbb{R}^d)$ say, and that

$$\forall\, x \in \mathbb{R}^d, \qquad \partial^\alpha R_P(x) = \mathbb{E}[(\partial^\alpha K)(x - Z)]$$

for any $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq j$, where Z is a random d-vector with law P. Let $g_\beta(x) := |R_P(x)|^2 - \beta^2$. Then $g_\beta \in \mathcal{C}^j(\mathbb{R}^d)$ since the map $z \mapsto |z|^2$ is smooth over $\mathbb{R}^d$. We obviously have that

$$\mathcal{C}_P^\beta = \{x \in \mathbb{R}^d : g_\beta(x) = 0\}.$$

Fix $z = (\tilde{z}, z_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$ be such that $z \in \mathcal{C}_P^\beta$ and assume that $\nabla g_\beta(z) \neq 0$. Then, the implicit function theorem entails that there exists an open neighbourhood $U \subset \mathbb{R}^{d-1}$ of $\tilde{z}$, an open neighbourhood $V \subset \mathbb{R}^d$ of z, and a map $\varphi \in \mathcal{C}^j(U, \mathbb{R})$ such that $\varphi(\tilde{z}) = z_d$ and

$$V \cap \mathcal{C}_P^\beta = \{(\tilde{x}, \varphi(\tilde{x})) : \tilde{x} \in U\}.$$

In other words, in a neighbourhood of z, $\mathcal{C}_P^\beta$ is the graph of a function of class $\mathcal{C}^j$, which proves the claim. It remains to show that $\nabla g_\beta(z) \neq 0$. Because $R_P \in \mathcal{C}^1(\mathbb{R}^d)$, we have

$$\nabla g_\beta(z) = 2 J_{R_P}(z)^T R_P(z),$$

where $J_{R_P}(z)^T$ stands for the transpose of the Jacobian matrix of R_P at z. Recall that $\partial_j R_P(z) = \mathbb{E}[(\partial_j K)(z - Z)]$ and that

$$\forall\, x \in \mathbb{R}^d \setminus \{0\}, \qquad J_K(x) = \frac{1}{|x|}\Big(I_d - \frac{x x^T}{|x|^2}\Big),$$

where I_d stands for the $d \times d$ identity matrix. Consequently, we have

$$J_{R_P}(z) = \mathbb{E}\bigg[\frac{1}{|z - Z|}\Big(I_d - \frac{(z - Z)(z - Z)^T}{|z - Z|^2}\Big)\mathbb{I}[Z \neq z]\bigg].$$

The matrix $J_{R_P}(z)$ is obviously symmetric and non-negative definite. Let us show that it is positive definite. Assume, ad absurdum, that there exists $v \in {}^{d-1}$ such that $v^T J_{R_P}(z) v = 0$, i.e.

$$\mathbb{E}\bigg[\frac{1}{|z - Z|}\Big(1 - \Big(v, \frac{z - Z}{|z - Z|}\Big)^2\Big)\mathbb{I}[Z \neq z]\bigg] = 0.$$

We then have
$$\frac{1}{|z-Z|}\Big(1-\Big(v,\frac{z-Z}{|z-Z|}\Big)^2\Big)\mathbb{I}[Z\neq z]=0$$
P-almost surely. Because P admits a density, we have $\frac{1}{|z-Z|}\mathbb{I}[Z\neq z]\neq 0$ with P-probability 1. Consequently, we have
$$\Big|\Big(v,\frac{z-Z}{|z-Z|}\Big)\Big|=1$$
with P-probability 1. This implies that P is supported on the line through z with direction v, a contradiction. We deduce that $J_{R_P}(z)$ is positive definite, hence invertible. It follows that $J_{R_P}(z)^T R_P(z)\neq 0$, whence $\nabla g_\beta(z)\neq 0$. This concludes the proof. ■

5.2. **Probability content.** Unlike center-outward quantiles based on optimal transport [7], geometric quantile regions are not indexed by their probability content, i.e. we do not have $P[\mathcal{D}_P^\beta]=\beta$ in general. However, one can in principle re-index quantile regions so that they match their probability content. Assume that P admits a density f_P over $\mathbb{R}^d$ such that $f_P(x)>0$ for any $x\in\mathbb{R}^d$. Let $\theta_P(\beta)=P[\mathcal{D}_P^\beta]$ for any $\beta\in[0,1)$. Because quantile regions are nested, the map θ_P is monotone non-decreasing. The assumptions on P further ensure that $\theta_P:[0,1)\to[0,1)$ is continuous and bijective. It follows that the re-indexed quantile regions
$$\widetilde{\mathcal{D}}_P^\beta:=\mathcal{D}_P^{\theta_P^{-1}(\beta)}$$
match their probability content, i.e. we have $P[\widetilde{\mathcal{D}}_P^\beta]=\beta$ for any $\beta\in[0,1)$. We similarly define the re-indexed quantile contours
$$\widetilde{\mathcal{C}}_P^\beta:=\mathcal{C}_P^{\theta_P^{-1}(\beta)}$$
for any $\beta\in[0,1)$. This suggests defining an alternative rank function $\widetilde{R}_P(x)$. To do so, observe that $x\in\widetilde{\mathcal{C}}_P^\beta$ if and only if
$$\Big|\frac{\beta}{\theta_P^{-1}(\beta)}R_P(x)\Big|=\beta.$$
When the previous equality holds, we have $\beta=\theta(|R_P(x)|)$. This suggests letting
$$\widetilde{R}_P(x)=\theta_P(|R_P(x)|)\frac{R_P(x)}{|R_P(x)|},\qquad\forall\, x\in\mathbb{R}^d.$$
We then have
$$\widetilde{\mathcal{D}}_P^\beta=\Big\{x\in\mathbb{R}^d:|\widetilde{R}_P(x)|\leq\beta\Big\}\quad\text{and}\quad\widetilde{\mathcal{C}}_P^\beta=\Big\{x\in\mathbb{R}^d:|\widetilde{R}_P(x)|=\beta\Big\}$$
for any $\beta\in[0,1)$. Letting Z denote a random d-vector with law P, it follows that $\theta_P(|R_P(Z)|)$ is uniformly distributed over $[0,1)$. Indeed, we have
$$\forall\,\beta\in[0,1),\qquad P\Big[\theta_P(|R_P(Z)|)\leq\beta\Big]=P\Big[|R_P(Z)|\leq\theta_P^{-1}(\beta)\Big]=P\big[\widetilde{\mathcal{D}}_P^\beta\big]=\beta.$$
In fact, θ_P is the cdf of $|R_P(Z)|$. Although this construction in principle allows one to re-index depth regions so that they match their probability content, it requires knowing θ_P, hence the distribution of $|R_P(Z)|$. In addition, any transformation $\phi:\mathbb{R}^d\to\mathbb{R}^d$ such that the distribution of $\phi(R_P(Z))$ is uniform over $[0,1)$ depends explicitly on P, and leads to a circle argument. This is a consequence of the fact that, irrespective of the support of

P, the map R_P spans the whole space $\mathbb{R}^d$; see, e.g., Theorem 6.2 in [14]. Consequently, it is hopeless to try to gain information on the probability content of depth regions through deth rank itself. Nevertheless, we established in Section 3 that

$$\forall\ x \in \mathbb{R}^d, \qquad f_P(x) = \gamma_d\ (-\Delta)^{\frac{d-1}{2}}(\nabla \cdot R_P)(x).$$

Because the density f_P contains all the information on probability contents, R_P also does in some sense. Interchanging the order of differential operators yields

$$\forall\ x \in \mathbb{R}^d, \qquad f_P(x) = \gamma_d\ \nabla \cdot \big((-\Delta)^{\frac{d-1}{2}} R_P\big)(x),$$

where $(-\Delta)^{\frac{d-1}{2}} R_P : \mathbb{R}^d \to \mathbb{R}^d$ is the vector field defined through

$$\forall\ i \in \{1, \ldots, d\}, \qquad ((-\Delta)^{\frac{d-1}{2}} R_P)_i = (-\Delta)^{\frac{d-1}{2}}\big((R_P)_i\big).$$

Consequently, the divergence theorem entails that for any (regular) open and bounded subset $\Omega \subset \mathbb{R}^d$, we have

$$P[\Omega] = \int_\Omega f_P(x)\, dx = \gamma_d \int_{\partial\Omega} \Big((-\Delta)^{\frac{d-1}{2}} R_P(x), \nu(x)\Big)\, d\mathcal{H}_{d-1}(x), \tag{5.1}$$

where $\mathcal{H}_{d-1}$ denotes the $(d-1)$-dimensional Hausdorff measure, and $\nu(x)$ is the outer unit normal vector to Ω at x. Therefore, the probability content of an a given region is controlled by $(-\Delta)^{\frac{d-1}{2}} R_P$. Therefore, letting P be a probability measure on $\mathbb{R}$ and F its cdf, (5.1) is a multivariate analog of the equality

$$\forall\ a < b, \qquad P\big[(a, b]\big] = F_P(b) - F_P(a).$$

Notice that $(-\Delta)^{\frac{d-1}{2}} R_P$ and R_P actually coincide when $d = 1$. They differ only when $d > 1$; in this setting, the concept of geometric rank generates a spectrum

$$\Big\{\partial^\alpha R_P : \alpha \in \mathbb{N}^d,\ |\alpha| \leq d-1\Big\}$$

of intermediate functions "between" $(-\Delta)^{\frac{d-1}{2}} R_P$ and R_P. The properties of F when $d = 1$ split across the derivatives $\partial^\alpha R_P$. For instance, $(-\Delta)^{\frac{d-1}{2}} R_P$ controls the probability content, while R_P is bounded by 1 and converges to 1 at infinity.

6. Localisation issues

In this section we investigate the local properties of the operator $\mathcal{L}_d$. The operator $\mathcal{L}_d = (-\Delta)^{\frac{d-1}{2}} \nabla \cdot$ displays substantially different behaviours in odd and even dimensions. This is due the nature of $(-\Delta)^{\frac{d-1}{2}}$, which depends on whether $\frac{d-1}{2}$ is an integer or not. When $\frac{d-1}{2} \in \mathbb{N}$, then $(-\Delta)^{\frac{d-1}{2}}$ is the classical differential operator that consists in applying the Laplacian $-\Delta$ successively $\frac{d-1}{2}$ times. This operator is local in nature : if smooth functions u_1 and u_2 coincide over an open subset $U \subset \mathbb{R}^d$, then $(-\Delta)^{\frac{d-1}{2}} u_1$ and $(-\Delta)^{\frac{d-1}{2}} u_2$ also coincide over U. When d is even, then $\frac{d-1}{2} \in \mathbb{R} \setminus \mathbb{N}$; in this case, we write $(-\Delta)^{\frac{d-1}{2}} = (-\Delta)^{1/2}(-\Delta)^{\frac{d-2}{2}}$. Although $(-\Delta)^{1/2}$ acts like a derivative in terms of regularity (see Proposition 2.6 in [27]), it is also known to be a non-local operator.

Multivariate geometric ranks characterise probability measures in arbitrary dimension d : if P and Q are Borel probability measures over $\mathbb{R}^d$ and if $R_P(x) = R_Q(x)$ for any $x \in \mathbb{R}^d$, then $P = Q$ (see Theorem 2.5 in [10]). When d is odd, we provide a refinement of this result in the next proposition, thanks to the local nature of $\mathcal{L}_d$.

Proposition 6.1. *Fix $d \geq 1$ be odd. Let P and Q be a Borel probability measures on $\mathbb{R}^d$. Let $\Omega \subset \mathbb{R}^d$ be an open subset, and assume that $R_P(x) = R_Q(x)$ for any $x \in \Omega$. Then, P and Q coincide over Ω, i.e. $P(E) = Q(E)$ for any Borel subset $E \subset \Omega$.*

Proof of Proposition 6.1. Theorem 3.3 entails that

$$\int_{\mathbb{R}^d} \psi(x)\, dP(x) = \int_{\mathbb{R}^d} (R_P(x), (\mathcal{L}_d^* \psi)(x))\, dx,$$

and

$$\int_{\mathbb{R}^d} \psi(x)\, dQ(x) = \int_{\mathbb{R}^d} (R_Q(x), (\mathcal{L}_d^* \psi)(x))\, dx$$

for any $\psi \in \mathcal{S}(\mathbb{R}^d)$. In particular, these equalities hold for any $\psi \in \mathcal{C}_c^\infty(\Omega)$. Because $\frac{d-1}{2}$ is an integer, $\mathcal{L}_d^* = \gamma_d\, \nabla(-\Delta^{\frac{d-1}{2}})$ is a (non-fractional) differential operator. In particular, $\mathcal{L}_d^* \psi$ is also supported in Ω. Because $R_P = R_Q$ over Ω, we have

$$\forall\, \psi \in \mathcal{C}_c^\infty(\Omega), \qquad \int_\Omega \psi(x)\, dP(x) = \int_\Omega \psi(x)\, dQ(x).$$

It follows that $P(E) = Q(E)$ for any Borel subset $E \subset \Omega$. ■

When d is even, the operator $\mathcal{L}_d$ is non-local. In particular, the proof of Proposition 6.1 does not apply. We present two approaches attempting to recover a localisation result similar to Proposition 6.1.

Fix d even, and consider a probability measure P on $\mathbb{R}^d$. The first idea that naturally comes to mind is to embed P into $\mathbb{R}^{d+1}$ (with $d+1$ odd); this gives rise to a probability measure P^* supported on the hyperplane $x_{d+1} = 0$ of $\mathbb{R}^{d+1}$. Proposition 6.1 now applies to P^*. The other approach consists in localising the operator $(-\Delta)^{1/2}$. For a smooth function u on $\mathbb{R}^d$, computing $(-\Delta)^{1/2} u$ can be achieved by first solving $-\Delta U = 0$ over $\mathbb{R}_+^{d+1} := \mathbb{R}^d \times (0, \infty)$ subject to the boundary condition $U(\tilde{x}, 0) = u(\tilde{x})$ for any $\tilde{x} \in \mathbb{R}^d$. Then, we have

$$\forall\, \tilde{x} \in \mathbb{R}^d, \qquad ((-\Delta)^{1/2} u)(\tilde{x}) = -\lim_{x_{d+1} \to 0} (\partial_{d+1} U)(\tilde{x}, x_{d+1}).$$

Because the values of $\partial_{d+1} U$ in some open subset $\Omega \subset \mathbb{R}_+^{d+1}$ depend on the values of U on Ω only, this formulation is local with respect to U. For further details on this method, we refer the reader to [33] and [27].

It turns out that both approaches are equivalent. This is the content of the next proposition, in which we will show that the density f_P of P can be recovered through $\lim_{x_{d+1} \to 0} \partial_{d+1} U(\tilde{x}, x_{d+1})$, where $U(\tilde{x}, x_{d+1})$ is essentially equal to $(-\Delta)^{\frac{d-2}{2}} (\nabla \cdot R_{P^*})(\tilde{x}, x_{d+1})$ and solves $-\Delta U = 0$ over $\mathbb{R}_+^{d+1}$.

Throughout, we denote $\mathcal{H}_d$ the d-dimensional Hausdorff measure on $\mathbb{R}^{d+1}$.

Proposition 6.2. *Let $d \geq 2$ be even and P a Borel probability measure on $\mathbb{R}^d$. Assume that P admits a density $f_P \in L^1(\mathbb{R}^d)$ with respect to the Lebesgue measure and that $f_P \in \mathcal{C}^{0,\alpha}(\mathbb{R}^d)$ for some $\alpha \in (0,1)$. Let P^* denote the probability measure on $\mathbb{R}^{d+1}$ supported on* *the hyperplane $x_{d+1} = 0$ with density f_P with respect to $\mathcal{H}_d$. Let Z be a random d-vector with law P and Z^* a random $(d+1)$-vector with law P^*. Define*

$$U(x) = 2\gamma_{d+1}\mathbb{E}\Big[((-\Delta)^{\frac{d-2}{2}}(\nabla \cdot K_{d+1}))(x - Z^*)\Big], \qquad \forall\, x \in \mathbb{R}^{d+1}_+ := \mathbb{R}^d \times (0, \infty),$$

and

$$u(\tilde{x}) = \gamma_d\, (-\Delta)^{\frac{d-2}{2}}(\nabla \cdot R_P)(\tilde{x}), \qquad \forall\, \tilde{x} \in \mathbb{R}^d.$$

Then $U \in \mathcal{C}^\infty(\mathbb{R}^{d+1}_+)$ and $u \in \mathcal{C}^1(\mathbb{R}^d)$. In addition, the following holds :

(1) $U(x) = 2\gamma_{d+1}\, (-\Delta)^{\frac{d-2}{2}}(\nabla \cdot R_{P^*})(x)$ and $-\Delta U(x) = 0$, for any $x \in \mathbb{R}^{d+1}_+$;

(2) for any $\tilde{x} \in \mathbb{R}^d$, $U(\tilde{x}, 0) = u(\tilde{x})$ and

$$f_P(\tilde{x}) = ((-\Delta)^{1/2}u)(\tilde{x}) = \lim_{x_{d+1} \overset{>}{\to} 0} -(\partial_{d+1}U)(\tilde{x}, x_{d+1}).$$

In practice, Proposition 6.2 entails that one can recover f_P by applying purely (local) differential operators to the geometric rank associated to P^* instead of P. We summarize this in the following corollary.

Corollary 6.3. *Let $d \geq 2$ be even and P a Borel probability measure on $\mathbb{R}^d$. Assume that P admits a density $f_P \in L^1(\mathbb{R}^d)$ with respect to the Lebesgue measure and that $f_P \in \mathcal{C}^{0,\alpha}(\mathbb{R}^d)$ for some $\alpha \in (0, 1)$. Let P^* denote the probability measure on $\mathbb{R}^{d+1}$ supported on the hyperplane $x_{d+1} = 0$ with density f_P with respect to $\mathcal{H}_d$. Then,*

$$\forall\, \tilde{x} \in \mathbb{R}^d, \qquad f_P(\tilde{x}) = -2\gamma_{d+1} \lim_{x_{d+1} \overset{>}{\to} 0} \partial_{d+1}(-\Delta)^{\frac{d-2}{2}}(\nabla \cdot R_{P^*})(\tilde{x}, x_{d+1}).$$

Proof of Proposition 6.2. Because P^* admits the null density over the open subset $\mathbb{R}^{d+1}_+ := \mathbb{R}^d \times (0, \infty)$ of $\mathbb{R}^{d+1}$, Proposition 3.4 entails that $R_{P^*} \in \mathcal{C}^d(\mathbb{R}^{d+1}_+)$ and that

$$\forall\, x \in \mathbb{R}^{d+1}_+, \qquad \partial^\alpha R_{P^*}(x) = \mathbb{E}[(\partial^\alpha K_{d+1})(x - Z^*)]$$

It follows that

$$\forall\, x \in \mathbb{R}^{d+1}_+, \qquad U(x) = 2\gamma_{d+1}\, (-\Delta)^{\frac{d-2}{2}}(\nabla \cdot R_{P^*})(x).$$

Theorem 3.5 further implies that $-\Delta U(x) = 0$ for any $x \in \mathbb{R}^{d+1}_+$. Let us show that $U(\tilde{x}, 0) = u(\tilde{x})$ for any $\tilde{x} \in \mathbb{R}^d$. Proposition 3.4 entails that $R_P \in \mathcal{C}^{d-1}(\mathbb{R}^d)$ and

$$\forall\, \tilde{x} \in \mathbb{R}^d, \qquad u(\tilde{x}) = \gamma_d\, \mathbb{E}\Big[((-\Delta)^{\frac{d-2}{2}}(\nabla \cdot K_d))(\tilde{x} - Z)\Big].$$

Let us compute explicitly $(-\Delta)^{\frac{d-2}{2}}(\nabla \cdot K_d)$ and $(-\Delta)^{\frac{d-2}{2}}(\nabla \cdot K_{d+1})$. It is easy to see that

$$\forall\, \tilde{x} \in \mathbb{R}^d \setminus \{0\}, \qquad (\nabla \cdot K_d)(\tilde{x}) = (d-1)\frac{1}{|\tilde{x}|}.$$

Easy computations further show that

$$\forall\, \tilde{x} \in \mathbb{R}^d \setminus \{0\}, \qquad (-\Delta)^\ell \frac{1}{|\tilde{x}|} = \Lambda_{d,\ell} \frac{1}{|\tilde{x}|^{2\ell+1}}, \tag{6.1}$$

where

$$\forall\, 1 \leq \ell \leq \frac{d-2}{2}, \qquad \Lambda_{d,\ell} = \prod_{j=1}^{\ell}(2j-1)(d-2j-1). \tag{6.2}$$

This provides

$$\forall\ \tilde{x} \in \mathbb{R}^d \setminus \{0\}, \qquad (-\Delta)^{\frac{d-2}{2}}(\nabla \cdot K_d)(\tilde{x}) = (d-1)\Lambda_{d,\frac{d-2}{2}} \frac{1}{|\tilde{x}|^{d-1}}$$

The same computations yield

$$\forall\ x \in \mathbb{R}^{d+1} \setminus \{0\}, \qquad (-\Delta)^{\frac{d-2}{2}}(\nabla \cdot K_{d+1})(x) = d\ \Lambda_{d+1,\frac{d-2}{2}} \frac{1}{|x|^{d-1}}.$$

Using the fact that $\prod_{j=1}^{k}(2j-1) = \frac{(2k)!}{2^k k!}$ for any integer $k \geq 1$, it is easy to see that

$$\Lambda_{d,\frac{d-2}{2}} = \left(\frac{\Gamma(d-1)}{2^{\frac{d-2}{2}}\Gamma(\frac{d}{2})}\right)^2$$

and $\Lambda_{d+1,\frac{d-2}{2}} = \Gamma(d-1)$. It follows that

$$\forall\ x \in \mathbb{R}^{d+1}_+, \qquad U(x) = 2\gamma_{d+1} d\ \Gamma(d-1)\mathbb{E}\left[\frac{1}{|x-Z^*|^{d-1}}\mathbb{I}[Z^* \neq x]\right]$$

and

$$\forall\ \tilde{x} \in \mathbb{R}^d, \qquad u(\tilde{x}) = \gamma_d\ (d-1)\left(\frac{\Gamma(d-1)}{2^{\frac{d-2}{2}}\Gamma(\frac{d}{2})}\right)^2 \mathbb{E}\left[\frac{1}{|\tilde{x}-Z|^{d-1}}\mathbb{I}[Z \neq \tilde{x}]\right].$$

In particular, we have

$$\begin{aligned}
\forall\ \tilde{x} \in \mathbb{R}^d, \qquad U(\tilde{x},0) &= 2d\ \gamma_{d+1}\ \Gamma(d-1)\mathbb{E}\left[\frac{1}{|\tilde{x}-Z|^{d-1}}\mathbb{I}[Z \neq \tilde{x}]\right] \\
&= 2d\ \gamma_{d+1}\ \Gamma(d-1) \times \frac{2^{d-2}\Gamma(\frac{d}{2})^2}{\gamma_d(d-1)\Gamma(d-1)^2} u(\tilde{x}) \\
&= \frac{\gamma_{d+1}}{\gamma_d} \times \frac{2^{d-1} d\Gamma(\frac{d}{2})^2}{\Gamma(d)} u(\tilde{x}).
\end{aligned}$$

Using the fact that

$$\Gamma\left(k+\frac{1}{2}\right) = \frac{\sqrt{\pi}\ \Gamma(2k+1)}{2^{2k}\Gamma(k+1)} \tag{6.3}$$

for any $k \in \mathbb{N}$ leads to

$$\frac{\gamma_{d+1}}{\gamma_d} = \frac{\Gamma(d+1)}{2^{d+1}\Gamma(\frac{d}{2}+1)^2} = \frac{d\ \Gamma(d)}{2^{d+1}(\frac{d}{2}\Gamma(\frac{d}{2}))^2} = \frac{\Gamma(d)}{2^{d-1} d\Gamma(\frac{d}{2})^2}.$$

It follows that $U(\tilde{x},0) = u(\tilde{x})$ for any $\tilde{x} \in \mathbb{R}^d$. Now, let us compute $-\partial_{d+1}U(\tilde{x}, x_{d+1})$ for any $(\tilde{x}, x_{d+1}) \in \mathbb{R}^d \times (0,\infty)$. We have already noticed that

$$\forall\ x \in \mathbb{R}^{d+1}_+, \qquad \partial_{d+1}U(x) = 2\gamma_{d+1}\mathbb{E}\left[\partial_{d+1}((-\Delta)^{\frac{d-2}{2}}(\nabla \cdot K_{d+1}))(x-Z^*)\right].$$

Writing $Z^* = (Z_1^*, \ldots, Z_{d+1}^*)$, we then have

$$\forall\ (\tilde{x}, x_{d+1}) \in \mathbb{R}^d \times (0,\infty), \qquad -(\partial_{d+1}U)(\tilde{x}, x_{d+1}) = 2\gamma_{d+1}\Gamma(d+1)\mathbb{E}\left[\frac{x_{d+1}-Z_{d+1}^*}{|x-Z^*|^{d+1}}\mathbb{I}[Z^* \neq x]\right]$$

Let us show that

$$\lim_{x_{d+1} \overset{>}{\to} 0} \mathbb{E}\left[\frac{x_{d+1}-Z_{d+1}^*}{|x-Z^*|^{d+1}}\mathbb{I}[Z \neq x]\right] = \frac{\pi^{\frac{d+1}{2}}}{\Gamma(\frac{d+1}{2})} f_P(\tilde{x}).$$

For any $x \in \mathbb{R}^{d+1}_+$, we have

$$\mathbb{E}\left[\frac{x_{d+1} - Z^*_{d+1}}{|x - Z^*|^{d+1}} \mathbb{I}[Z^* \neq x]\right] = \int_{\{z_{d+1}=0\}} \frac{x_{d+1} - z_{d+1}}{\left(|\tilde{x} - \tilde{z}|^2 + (x_{d+1} - z_{d+1})^2\right)^{\frac{d+1}{2}}} f_P(\tilde{z})\, d\mathcal{H}_d(\tilde{z}, z_{d+1})$$

$$= \int_{\mathbb{R}^d} \frac{x_{d+1}}{(|\tilde{x} - \tilde{z}|^2 + x_{d+1}^2)^{\frac{d+1}{2}}} f_P(\tilde{z})\, d\tilde{z}$$

$$= \int_{\mathbb{R}^d} \frac{1}{x_{d+1}^d} \frac{1}{\left(1 + |\frac{\tilde{x}-\tilde{z}}{x_{d+1}}|^2\right)^{\frac{d+1}{2}}} f_P(\tilde{z})\, d\tilde{z}$$

$$= \int_{\mathbb{R}^d} \frac{1}{(1 + |\tilde{z}|^2)^{\frac{d+1}{2}}} f_P(\tilde{x} - x_{d+1}\tilde{z})\, d\tilde{z}.$$

We have

$$\left| \int_{\mathbb{R}^d} \frac{1}{(1 + |\tilde{z}|^2)^{\frac{d+1}{2}}} f_P(\tilde{x} - x_{d+1}\tilde{z}) d\tilde{z} - f_P(\tilde{x}) \int_{\mathbb{R}^d} \frac{1}{(1 + |\tilde{z}|^2)^{\frac{d+1}{2}}}\, d\tilde{z} \right|$$
$$\leq\ [f_P]_{\mathcal{C}^{0,\alpha}} |x_{d+1}|^\alpha \int_{\mathbb{R}^d} \frac{|\tilde{z}|^\alpha}{(1 + |\tilde{z}|^2)^{\frac{d+1}{2}}}\, d\tilde{z},$$

where $[f_P]_{\mathcal{C}^{0,\alpha}} := \sup_{x \neq y} \frac{|f_P(x) - f_P(y)|}{|x-y|^\alpha}$. Because $\alpha < 1$, we have

$$\int_{\mathbb{R}^d} \frac{|\tilde{z}|^\alpha}{(1 + |\tilde{z}|^2)^{\frac{d+1}{2}}}\, d\tilde{z} < \infty.$$

Because $\alpha > 0$, it follows that

$$\lim_{x_{d+1} \overset{>}{\to} 0} \mathbb{E}\left[\frac{x_{d+1} - Z^*_{d+1}}{|x - Z^*|^{d+1}} \mathbb{I}[Z^* \neq x]\right] = f_P(\tilde{x}) \int_{\mathbb{R}^d} \frac{1}{(1 + |\tilde{z}|^2)^{\frac{d+1}{2}}}\, d\tilde{z}.$$

Furthermore, one can show that

$$\int_{\mathbb{R}^d} \frac{1}{(1 + |\tilde{z}|^2)^{\frac{d+1}{2}}}\, d\tilde{z} = S_{d-1} \frac{\sqrt{\pi}\ \Gamma(d/2)}{2\Gamma(\frac{d+1}{2})} = \frac{\pi^{\frac{d+1}{2}}}{\Gamma(\frac{d+1}{2})},$$

where $S_{d-1} = 2\pi^{\frac{d}{2}}/\Gamma(\frac{d}{2})$ is the surface area of the $(d-1)$-dimensional sphere of $\mathbb{R}^d$. We deduce that

$$\lim_{x_{d+1} \overset{>}{\to} 0} -(\partial_{d+1} U)(\tilde{x}, x_{d+1}) = 2\gamma_{d+1}\Gamma(d+1) \frac{\pi^{\frac{d+1}{2}}}{\Gamma(\frac{d+1}{2})} f_p(\tilde{x}).$$

Using again (6.3), we see that

$$2\gamma_{d+1}\Gamma(d+1) \frac{\pi^{\frac{d+1}{2}}}{\Gamma(\frac{d+1}{2})} = 1.$$

It follows that

$$\lim_{x_{d+1} \overset{>}{\to} 0} -(\partial_{d+1} U)(\tilde{x}, x_{d+1}) = f_P(\tilde{x}).$$

Because $f \in \mathcal{C}^{0,\alpha}(\mathbb{R}^d)$, Theorem 3.6 entails that

$$\forall\, \tilde{x} \in \mathbb{R}^d, \qquad f_P(\tilde{x}) = \gamma_d\, (-\Delta)^{1/2}(-\Delta)^{\frac{d-2}{2}}(\nabla \cdot R_P)(\tilde{x}) = ((-\Delta)^{1/2}u)(\tilde{x}).$$

This concludes the proof. ■

7. Perspectives for future research

We showed that in any Euclidean space $\mathbb{R}^d$, an arbitrary probability measure P can be recovered explicitly through the partial differential equation $P = \mathcal{L}_d(R_P)$, where $\mathcal{L}_d$ is a (potentially fractional) linear differential operator. Eventhough this construction is given in closed form, the fact that the operator $\mathcal{L}_d$ is local when d is odd, and non-local when d is even is quite suprising. This leads to fundamentally different reconstruction procedures when applied to specific examples.

Although one couldn't have guessed that geometric quantiles and ranks can control the probability content of the corresponding depth regions, we showed that this is actually the case, through the derivatives of order $d - 1$ of the geometric rank. However, this does not feel very natural for a quantile concept, as we would expect that the rank itself control the probability content.

We feel that this paper exhibited the limitations of the concept of geometric quantiles and ranks through, among others, the severe duality between odd and even dimensions. Nevertheless, our work calls upon potential generalisations of the framework developed in the present paper. Indeed, the nature of the PDE we established relies entirely on the kernel K. For instance, it would be natural to look for other kernels K, therefore leading to different reconstruction procedures, and study the resulting properties of the corresponding rank concepts.

Acknowledgements

The author thanks Davy Paindaveine for the many comments, suggestions, and the help he gave for the realisation of this work. The author also thanks Stanislav Nagy and Yvik Swan for their precious comments on this work, as well as the F.R.S.-FNRS for financial support.

References

[1] J. W. Tukey, "Mathematics and the picturing of data," in *Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974), Vol. 2*, pp. 523–531, Canad. Math. Congress, Montreal, Que., 1975.

[2] R. Y. Liu, "On a notion of data depth based on random simplices," *Ann. Statist.*, vol. 18, no. 1, pp. 405–414, 1990.

[3] R. Y. Liu, J. M. Parelius, and K. Singh, "Multivariate analysis by data depth: descriptive statistics, graphics and inference (with discussion)," *Ann. Statist.*, vol. 27, no. 3, pp. 783–858, 1999.

[4] Y. Vardi and C.-H. Zhang, "The multivariate L_1-median and associated data depth," *Proc. Natl. Acad. Sci. USA*, vol. 97, no. 4, pp. 1423–1426, 2000.

[5] Y. Zuo, "Projection-based depth functions and associated medians," *Ann. Statist.*, vol. 31, no. 5, pp. 1460–1490, 2003.

[6] M. Hallin, D. Paindaveine, and M. Šiman, "Multivariate quantiles and multiple-output regression quantiles: From L_1 optimization to halfspace depth (with discussion)," *Ann. Statist.*, vol. 38, pp. 635–669, 2010.

[7] M. Hallin, E. del Barrio, J. C. Cuesta-Albertos, and C. Matrán, "Distribution and quantile functions, ranks and signs in dimension d: a measure transportation approach," *Ann. Statist.*, vol. 49, pp. 1139–1165, 2021.
[8] R. Serfling, "Quantile functions for multivariate analysis: approaches and applications," *Stat. Neerl.*, vol. 56, pp. 214–232, 2002.
[9] P. Chaudhuri, "On a geometric notion of quantiles for multivariate data," *J. Amer. Statist. Assoc.*, vol. 91, pp. 862–872, 1996.
[10] V. I. Koltchinski, "M-estimation, convexity and quantiles," *Ann. Statist.*, vol. 25, pp. 435–477, 1997.
[11] A. Chakraborty and P. Chaudhuri, "The spatial distribution in infinite dimensional spaces and related quantiles and depths," *Ann. Statist.*, vol. 42, pp. 1203–1231, 2014.
[12] J. Chowdhury and P. Chaudhuri, "Nonparametric depth and quantile regression for functional data," *Bernoulli*, vol. 25, pp. 395–423, 2019.
[13] H. Oja, *Multivariate Nonparametric Methods with R.* Springer, 2010.
[14] D. Konen and D. Paindaveine, "Multivariate ρ-quantiles: a spatial approach," *Bernoulli,*, vol. 28, pp. 1912–1934, 2022.
[15] T. Ferguson, *Mathematical Statistics : A decision theoretic approach.* Academic Press, 1967.
[16] D. Paindaveine and J. Virta, "On the behavior of extreme d-dimensional spatial quantiles under minimal assumptions," in *Advances in Contemporary Statistics and Econometrics* (A. Daouia and A. Ruiz-Gazen, eds.), (Cham.), pp. 243–259, Springer, 2020.
[17] S. Girard and G. Stupfler, "Extreme geometric quantiles in a multivariate regular variation framework," *Extremes*, vol. 18, pp. 629–663, 2015.
[18] S. Girard and G. Stupfler, "Intriguing properties of extreme geometric quantiles," *REVSTAT*, vol. 15, pp. 107–139, 2017.
[19] S. Nagy, "Halfspace depth does not characterize probability distributions," *Statistical Papers*, 2019.
[20] A. Struyf and P. Rousseeuw, "Halfspace depth and regression depth characterize the empirical distribution," *Journal of Multivariate Analysis*, vol. 69, no. 1, pp. 135–153, 1999.
[21] S. Nagy, "The halfspace depth characterization problem," in *Nonparametric statistics, 4th ISNPS* (Springer, ed.), 2020.
[22] W. Rudin, *Functional analysis.* McGraw-Hill, Inc., second ed., 1991.
[23] L. Evans, *Partial Differential Equations*, vol. 19. Amer. Math. Soc., 1998.
[24] X. Fernandez-Real and X. Ros-Oton, *Regularity theory for elliptic PDE.* Submitted, 2020.
[25] A. Lischke, G. Pang, M. Gulian, F. Song, C. Glusa, X. Zheng, Z. Mao, W. Cai, M. Meerschaert, M. Ainsworth, and G. Em Karniadakis, "What is the fractional Laplacian? A comparative review with new results," *J. Comput. Phys.*, 2020.
[26] E. Nezza, G. Palatucci, and E. Valdinoci, "Hitchhiker's guide to the fractional sobolev spaces," *Bulletin Des Sciences Mathematiques*, 2011.
[27] L. Silvestre, "Regularity of the obstacle problem for a fractional power of the Laplace operator," *Communications on Pure and Applied Mathematics*, 2007.
[28] P. Stinga, "User's guide to the fractional Laplacian and the method of semigroups," *Fractional Differential Equations*, 2018.
[29] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, *NIST Handbook of Mathematical Functions.* Cambridge University Press, 2010.
[30] A. P. Prudnikov, Y. A. Brychkov, and O. I. Marichev, *Integrals and Series: More Special Functions, Vol. 3.* New York: Gordon and Breach Science Publishers, 1990.
[31] F. Oberhettinger, *Tables of Bessel Transforms.* Berlin Heidelberg: Springer-Verlag, 1972.
[32] S. Nagy, "Monotonicity properties of spatial depth," *Statistics and Probability Letters*, 2017.
[33] L. Caffarelli and L. Silvestre, "An extension problem related to the fractional Laplacian," *Communications in Partial Differential Equations*, 2007.
[34] N. S. Landkof, *Foundations of Modern Potential Theory.* Springer-Verlag, 1972.
[35] S. Bochner, *Lectures on Fourier Integrals.* Princeton University Press, 1959.
[36] F. Olver, D. Lozier, R. Boisvert, and C. Clark, eds., *NIST Handbook of Mathematical Functions.* National Institute of Standards and Technology, and Cambridge University Press, 2010.
[37] V. I. Bogachev, *Weak Convergence of Measures.* Amer. Math. Soc., 2018.

Appendix A. Appendix

A.1. **Auxiliary proofs for Section 2.2.** *Proof of Corollary 2.8.* Let $x_0 \in \Omega$ and $r > 0$ be such that $\overline{B(x_0, r)} \subset \Omega$. For any $x \in B_1$, let $\tilde{u}(x) := u(\frac{x-x_0}{r})$ and $\tilde{f}(x) := \frac{1}{r^2} f(\frac{x-x_0}{r})$. Since $\Delta u = f$ in the weak sense in $B(x_0, r)$, a direct computation entails that $\Delta \tilde{u} = \tilde{f}$ in the weak sense in B_1. Since $u \in H^1(B(x_0, r)) \cap L^\infty(B(x_0, r))$ and $f \in \mathcal{C}^{k,\alpha}(B(x_0, r))$, we have $\tilde{u} \in H^1(B_1) \cap L^\infty(B_1)$, and $\tilde{f} \in \mathcal{C}^{k,\alpha}(B_1)$. It follows from Proposition 2.7 that $\tilde{u} \in \mathcal{C}^{k+2,\alpha}(B_1)$. This implies that $u \in \mathcal{C}^{k+2,\alpha}(B(x_0, r))$. Now let $V \subset \Omega$ be an open subset such that $\overline{V} \subset \Omega$. Then V can be covered by a finite number of balls of the form $B(x_0, r)$ with $\overline{B(x_0, r)} \subset \Omega$. Since u is of class $\mathcal{C}^{k+2,\alpha}$ on each one of these balls, we have $u \in \mathcal{C}^{k+2,\alpha}(V)$. We conclude that $u \in \mathcal{C}^{k+2,\alpha}_{\text{loc}}(\Omega)$. ■

A.2. **Auxiliary proofs for Section 2.3.** *Proof of Proposition 2.9.* Observe that the map $\xi \mapsto (1 + |\xi|)^m |\xi|^{2s} (\mathcal{F}u)(\xi)$ is integrable over $\mathbb{R}^d$ for any $m \geq 0$ since $u \in \mathcal{S}(\mathbb{R}^d)$. This entails that $\mathcal{F}^{-1}(|\xi|^{2s} \mathcal{F}u) \in \mathcal{C}^m(\mathbb{R}^d)$ for any $m \geq 0$, whence $(-\Delta)^s u \in \mathcal{C}^\infty(\mathbb{R}^d)$. It remains to show that (2.4) holds for any $\alpha \in \mathbb{N}^d$. Observe that

$$\begin{aligned} \partial^\alpha (-\Delta)^s u &= \partial^\alpha \mathcal{F}^{-1}(|\xi|^{2s} \hat{u}) \\ &= \mathcal{F}^{-1}\Big((2i\pi\xi)^\alpha |\xi|^{2s} \mathcal{F}u\Big) \\ &= \mathcal{F}^{-1}\Big(|\xi|^{2s} \mathcal{F}(\partial^\alpha u)\Big) \\ &= (-\Delta)^s (\partial^\alpha u) \end{aligned}$$

for any $\alpha \in \mathbb{N}^d$. Since $\partial^\alpha u$ belongs to $\mathcal{S}(\mathbb{R}^d)$ for any $\alpha \in \mathbb{N}^d$, it is enough to show that

$$\sup_{x \in \mathbb{R}^d} |(1 + |x|^{d+2s})(-\Delta)^s u(x)| \lesssim |u|_{L^1(\mathbb{R}^d)} + \sup_{z \in \mathbb{R}^d} \Big((1 + |z|)^{d+2} |\nabla^2 u(z)|\Big). \tag{A.1}$$

By simple changes of variable, it is easy to show that

$$(-\Delta)^s u(x) = -\frac{1}{2} c_{d,s} \int_{\mathbb{R}^d} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{d+2s}} \, dy$$

for any $x \in \mathbb{R}^d$; see, e.g., Lemma 3.2 in [26]. Notice that this last integral is not singular at $y = 0$ anymore. Indeed, one can easily show that

$$\frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{d+2s}} = \frac{1}{|y|^{d+2s}} \int_{-1}^{1} (y, \nabla^2 u(x+ty)y) \, dt. \tag{A.2}$$

The r.h.s. of (A.2) is then bounded by

$$\frac{|\nabla^2 u|_{L^\infty(\mathbb{R}^d)}}{|y|^{d+2s-2}},$$

which is integrable near the origin. Let us first show that

$$\sup_{x \in \mathbb{R}^d} |(-\Delta)^s u(x)| \lesssim |u|_{L^1(\mathbb{R}^d)} + \sup_{z \in \mathbb{R}^d} \Big((1 + |z|)^{d+2} |\nabla^2 u(z)|\Big). \tag{A.3}$$

Let us fix $x \in \mathbb{R}^d$ and write

$$-\frac{2}{c_{d,s}}(-\Delta)^s u(x) = \int_{\mathbb{R}^d\setminus B_1} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{d+2s}}\,dy + \int_{B_1} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{d+2s}}\,dy =: I_1(x) + I_2(x).$$

For I_1, we have

$$|I_1(x)| \le 4|u|_{L^1(\mathbb{R}^d)}.$$

For I_2, recalling (A.2), we have

$$|I_2(x)| \le \int_{B_1} \frac{1}{|y|^{d+2s-2}} \int_{-1}^{1} |\nabla^2 u(x+ty)|\,dt\,dy \lesssim \sup_{z\in\mathbb{R}^d} |\nabla^2 u(z)| \le \sup_{z\in\mathbb{R}^d} \Big((1+|z|)^{d+2}|\nabla^2 u(z)|\Big).$$

This yields (A.3). Let us now show that

$$\sup_{x\in\mathbb{R}^d} \Big(|x|^{d+2s}|(-\Delta)^s u(x)|\Big) \lesssim |u|_{L^1(\mathbb{R}^d)} + \sup_{z\in\mathbb{R}^d}(1+|z|)^{d+2}|\nabla^2 u(z)|. \tag{A.4}$$

Let us fix $x \in \mathbb{R}^d$ and write

$$-\frac{2}{c_{d,s}}|x|^{d+2s}(-\Delta)^s u(x) = |x|^{d+2s}\int_{\mathbb{R}^d\setminus B_{\frac{1}{2}|x|}} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{d+2s}}\,dy + |x|^{d+2s}\int_{B_{\frac{1}{2}|x|}} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{d+2s}}\,dy =: J_1(x) + J_2(x).$$

For J_1, we have

$$|J_1(x)| \le 4|u|_{L^1(\mathbb{R}^d)}2^{d+2s}.$$

For J_2, recalling (A.2), we have

$$|J_2(x)| \le |x|^{d+2s}\int_{B_{\frac{1}{2}|x|}} \frac{1}{|y|^{d+2s-2}} \int_{-1}^{1} |\nabla^2 u(x+ty)|\,dt\,dy = \int_{B_{\frac{1}{2}|x|}} \frac{1}{|y|^{d+2s-2}} \int_{-1}^{1} \left(\frac{|x|}{|x+ty|}\right)^{d+2s} |x+ty|^{d+2s}|\nabla^2 u(x+ty)|\,dt\,dy.$$

For any $t \in [-1,1]$ and y with $|y| \le \frac{1}{2}|x|$, we have

$$\frac{|x|}{|x+ty|} \le \frac{|x|}{|x|-|t||y|} \le 2.$$

For any k, let

$$C_k(u) := \sup_{x\in\mathbb{R}^d} (1+|z|)^k |\nabla^2 u(z)| < \infty.$$

We then have

$$|x+ty|^{d+2s}|\nabla^2 u(x+ty)| \le \frac{C_{N+d+2s}(u)}{(1+|x+ty|)^N} \le \frac{C_{N+d+2s}(u)}{(1+\frac{1}{2}|x|)^N}$$

for any N, $|y| \le \frac{1}{2}|x|$, and $t \in [-1,1]$. Let us fix $N = 2-2s$. We then have

$$|J_2(x)| \le 2^{d+2s} C_{d+2}(u) \frac{1}{(1+\frac{1}{2}|x|)^{2-2s}} \int_{B_{\frac{1}{2}|x|}} \frac{1}{|y|^{d+2s-2}}\, dy.$$

Furthermore, it is easy to see that

$$\int_{B_{\frac{1}{2}|x|}} \frac{1}{|y|^{d+2s-2}}\, dy \lesssim |x|^{2-2s}.$$

It follows that

$$\sup_{x\in\mathbb{R}^d} |J_2(x)| \lesssim \sup_{x\in\mathbb{R}^d} (1+|z|)^{d+2} |\nabla^2 u(z)|.$$

We deduce that

$$\sup_{x\in\mathbb{R}^d} \Big(|x|^{d+2s} |(-\Delta)^s u(x)| \Big) \lesssim |u|_{L^1(\mathbb{R}^d)} + \sup_{x\in\mathbb{R}^d} (1+|z|)^{d+2} |\nabla^2 u(z)|,$$

which establishes (A.4). Putting (A.3) and (A.4) together yields (A.1), which concludes the proof. ■

Proof of Proposition 2.12. 1. Because $u \in L^2(\mathbb{R}^d)$, we have $u \in \mathcal{S}_s(\mathbb{R}^d)'$. In particular, $(-\Delta)^s u$ is a well-defined tempered distribution. Fix $\psi \in \mathcal{S}(\mathbb{R}^d)$. We have

$$\langle (-\Delta)^s u, \psi \rangle = \langle u, (-\Delta)^s \psi \rangle = (2\pi)^{2s} \int_{\mathbb{R}^d} u(x) \mathcal{F}^{-1}(|\xi|^{2s} \mathcal{F}\psi)(x)\, dx.$$

Because $u \in L^2(\mathbb{R}^d)$ and $|\xi|^{2s}\mathcal{F}\psi \in L^2(\mathbb{R}^d)$, we have

$$\langle (-\Delta)^s u, \psi \rangle = (2\pi)^{2s} \int_{\mathbb{R}^d} (\mathcal{F}^{-1}u)(\xi) |\xi|^{2s} (\mathcal{F}\psi)(\xi)\, d\xi$$

by interchanging the inverse Fourier transform under the integral. Since $u \in H^{2s}(\mathbb{R}^d)$, we have $|\xi|^{2s}\mathcal{F}^{-1}u \in L^2(\mathbb{R}^d)$. Since $\psi \in L^2(\mathbb{R}^d)$, interchanging the Fourier transform again yields

$$\langle (-\Delta)^s u, \psi \rangle = (2\pi)^{2s} \int_{\mathbb{R}^d} \mathcal{F}(|\xi|^{2s}(\mathcal{F}^{-1}u))(x)\psi(x)\, dx.$$

Observing that

$$\mathcal{F}(|\xi|^{2s}(\mathcal{F}^{-1}u)) = \mathcal{F}^{-1}(|\xi|^{2s}(\mathcal{F}u))$$

yields

$$\langle (-\Delta)^s u, \psi \rangle = \int_{\mathbb{R}^d} (2\pi)^{2s} \mathcal{F}^{-1}(|\xi|^{2s}(\mathcal{F}u))(x)\psi(x)\, dx.$$

Because we chose $\psi \in \mathcal{S}(\mathbb{R}^d)$ arbitrarily, the conclusion follows.

2. Observe that $|\xi|^{2s}\mathcal{F}u \in \mathcal{S}(\mathbb{R}^d)'$ since $|\xi|^{2s}\mathcal{F}u \in L^1(\mathbb{R}^d)$. In particular, $\mathcal{F}^{-1}(|\xi|^{2s}\mathcal{F}u) \in \mathcal{S}(\mathbb{R}^d)'$ as well. Let $\psi \in \mathcal{S}(\mathbb{R}^d)$. We have

$$\begin{aligned}\big\langle (2\pi)^{2s}\mathcal{F}^{-1}(|\xi|^{2s}\mathcal{F}u), \psi\big\rangle &= \big\langle |\xi|^{2s}\mathcal{F}u, (2\pi)^{2s}\mathcal{F}^{-1}\psi\big\rangle \\ &= \int_{\mathbb{R}^d} |\xi|^{2s}\mathcal{F}u(\xi)(2\pi)^{2s}(\mathcal{F}^{-1}\psi)(\xi)\, d\xi \\ &= \int_{\mathbb{R}^d} \mathcal{F}u(\xi)\mathcal{F}^{-1}\Big((2\pi)^{2s}\mathcal{F}\big(|\xi|^{2s}\mathcal{F}^{-1}\psi\big)\Big)(\xi)\, d\xi \\ &= \int_{\mathbb{R}^d} \mathcal{F}u(\xi)\mathcal{F}^{-1}((-\Delta)^s\psi)(\xi)\, d\xi.\end{aligned}$$

Because $\psi \in \mathcal{S}(\mathbb{R}^d)$, we have $(-\Delta)^s\psi \in \mathcal{S}_s(\mathbb{R}^d)$. In particular, $(-\Delta)^s\psi \in L^1(\mathbb{R}^d)$. Since $\mathcal{F}u \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $|\xi|^{2s}\mathcal{F}u$, it is clear that $\mathcal{F}u \in L^1(\mathbb{R}^d)$. By hat skipping, we have

$$\big\langle (2\pi)^{2s}\mathcal{F}^{-1}(|\xi|^{2s}\mathcal{F}u), \psi\big\rangle = \int_{\mathbb{R}^d} \mathcal{F}^{-1}(\mathcal{F}u)(x)((-\Delta)^s\psi)(x)\, dx.$$

Observe that since the tempered distribution $\mathcal{F}u$ is a function of $L^1(\mathbb{R}^d)$, we have $\mathcal{F}^{-1}(\mathcal{F}u)$ is a continuous function. It is further easy to see that $\mathcal{F}^{-1}(\mathcal{F}u) = u$ almost everywhere on $\mathbb{R}^d$, although u might not be integrable. We conclude that

$$\forall\ \psi \in \mathcal{S}(\mathbb{R}^d), \qquad \big\langle (2\pi)^{2s}\mathcal{F}^{-1}(|\xi|^{2s}\mathcal{F}u), \psi\big\rangle = \int_{\mathbb{R}^d} u(x)((-\Delta)^s\psi)(x)\, dx = \langle (-\Delta)^s u, \psi\rangle\, .$$

3. This is proved in [27]; see Proposition 2.4. ∎

A.3. **Auxiliary proofs for Section 3.** The proof of Theorem 3.2 requires the next two lemmas.

Lemma A.1. *Let $d \geq 2$ be an integer and $\alpha \in (0, d)$ a real number. Then the Fourier transform of the tempered distribution $1/|x|^\alpha$ is given by*

$$\mathcal{F}\Big(\frac{1}{|x|^\alpha}\Big)(\xi) = \frac{\Gamma(\frac{d-\alpha}{2})}{\pi^{\frac{d}{2}-\alpha}\Gamma(\frac{\alpha}{2})}\frac{1}{|\xi|^{d-\alpha}}.$$

Proof of Lemma A.1. The proof follows the same lines as [34] and [35] (see Equation (1.1.1) in [34] and Theorem 56 in [35]). Let $g \in L^1(\mathbb{R}^d)$ be such that $g(x) = h(|x|)$ for any $x \in \mathbb{R}^d$. We first prove that

$$\forall\ \xi \in \mathbb{R}^d \setminus \{0\}, \qquad \widehat{g}(\xi) = \frac{1}{(2\pi)^{\frac{d}{2}}|\xi|^d}\int_0^\infty r^{\frac{d}{2}} h\Big(\frac{r}{2\pi|\xi|}\Big) J_{\frac{d-2}{2}}(r)\, dr, \tag{A.5}$$

where J_ν is the Bessel function of the first kind of order ν. Fix $\xi \in \mathbb{R}^d \setminus \{0\}$. Because $g(x) = h(|x|)$ for any x, we have

$$\widehat{g}(\xi) = \int_{\mathbb{R}^d} g(x)e^{-2i\pi(x,\xi)}\, dx = \int_{\mathbb{R}^d} g(x)e^{-2i\pi(x,O\xi)}\, dx$$

for any $d \times d$ orthogonal matrix. Then assume that $\xi = |\xi|(1, 0, \dots, 0)$, and compute

$$\begin{aligned}\int_{\mathbb{R}^d} g(x) e^{-2i\pi(x,\xi)}\, dx &= \int_{\mathbb{R}^d} h(|x|) e^{-2i\pi|\xi|x_1}\, dx \\ &= \int_{\mathbb{R}} e^{-2i\pi|\xi|x_1} \left(\int_{\mathbb{R}^{d-1}} h\Big(\sqrt{x_1^2 + |y|^2}\Big)\, dy \right) dx_1 \\ &= S_{d-2} \int_{\mathbb{R}} e^{-2i\pi|\xi|x_1} \left(\int_0^\infty t^{d-2} h\Big(\sqrt{x_1^2 + t^2}\Big)\, dt \right) dx_1,\end{aligned}$$

where $S_{d-2} = 2\pi^{\frac{d-1}{2}}/\Gamma(\frac{d-1}{2})$ is the surface area of the $(d-2)$-dimensional unit sphere in $\mathbb{R}^{d-1}$. Write (x_1, t) in the spherical coordinates of the plane

$$(x_1, t) = (r\cos\theta, r\sin\theta),$$

with $r \in [0, \infty)$ and $\theta \in [0, \pi]$, since $t > 0$. This provides

$$\begin{aligned}\widehat{g}(\xi) &= S_{d-2} \int_0^\infty \left(\int_0^\pi e^{-2i\pi|\xi|r\cos\theta} (r\sin\theta)^{d-2} h(r)\, r d\theta \right) dr \\ &= S_{d-2} \int_0^\infty r^{d-1} h(r) \left(\int_0^\pi e^{-2i\pi|\xi|r\cos\theta} (\sin\theta)^{d-2} d\theta \right) dr.\end{aligned}$$

Now express $\int_0^\pi e^{-2i\pi|\xi|r\cos\theta}(\sin\theta)^{d-2}d\theta$ in terms of Bessel functions. For any $\nu \in \mathbb{C}$ with $\mathrm{Re}(\nu) > -1/2$, the Bessel function of the first kind of order ν satisfies

$$\begin{aligned}\forall\, x \in \mathbb{R}, \qquad J_\nu(x) &= \frac{(\frac{x}{2})^\nu}{\sqrt{\pi}\,\Gamma(\nu + \frac{1}{2})} \int_{-1}^1 (1 - t^2)^{\nu - \frac{1}{2}} \cos(xt)\, dt \\ &= \frac{(\frac{x}{2})^\nu}{\sqrt{\pi}\,\Gamma(\nu + \frac{1}{2})} \int_{-1}^1 (1 - t^2)^{\nu - \frac{1}{2}} e^{-ixt}\, dt;\end{aligned}$$

see (10.9.4) in [36]). Substituting $t = \cos\theta$ leads to

$$\forall\, x \in \mathbb{R}, \qquad J_\nu(x) = \frac{(\frac{x}{2})^\nu}{\sqrt{\pi}\,\Gamma(\nu + \frac{1}{2})} \int_0^\pi (\sin\theta)^{2\nu} e^{-ix\cos\theta}\, d\theta.$$

It follows that

$$\forall\, r > 0, \qquad \int_0^\pi e^{-2i\pi|\xi|r\cos\theta} (\sin\theta)^{d-2} d\theta = \frac{\sqrt{\pi}\,\Gamma(\frac{d-1}{2})}{(\pi|\xi|r)^{\frac{d-2}{2}}} J_{\frac{d-2}{2}}(2\pi|\xi|r).$$

We deduce that

$$\begin{aligned}\forall\, \xi \in \mathbb{R}^d \setminus \{0\}, \qquad \widehat{g}(\xi) &= \frac{2\pi}{|\xi|^{\frac{d-2}{2}}} \int_0^\infty r^{\frac{d}{2}} h(r) J_{\frac{d-2}{2}}(2\pi|\xi|r)\, dr \\ &= \frac{1}{(2\pi)^{\frac{d}{2}} |\xi|^d} \int_0^\infty r^{\frac{d}{2}} h\Big(\frac{r}{2\pi|\xi|}\Big) J_{\frac{d-2}{2}}(r)\, dr.\end{aligned}$$

This yields (A.5). Let $\alpha \in (0, \frac{d+1}{2})$. For any $k \in \mathbb{N}$, let $g_{k,\alpha}(x) := |x|^{\alpha-n}\mathbb{I}[0 < |x| < k]$ for any $x \in \mathbb{R}^d$, and $h_{k,\alpha}(t) = t^{\alpha-n}\mathbb{I}[0 < t < k]$ for any $t > 0$ so that $g_{k,\alpha}(x) = h_{k,\alpha}(|x|)$ for any $x \in \mathbb{R}^d$. For any k, we have $g_{k,\alpha} \in L^1(\mathbb{R}^d)$. For any $\xi \in \mathbb{R}^d$, applying (A.5) to $g_{k,\alpha}$ leads to

$$\widehat{g_{k,\alpha}}(\xi) = \frac{1}{(2\pi)^{\alpha-\frac{d}{2}}|\xi|^\alpha}\int_0^{2\pi|\xi|k} r^{\alpha-\frac{d}{2}} J_{\frac{d-2}{2}}(r)\,dr.$$

According to (10.22.43) in [36], this integral converges to

$$\int_0^\infty r^{\alpha-\frac{d}{2}} J_{\frac{d-2}{2}}(r)\,dr = 2^{\alpha-\frac{d}{2}}\frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{d-\alpha}{2})}$$

as $k \to \infty$, since $\alpha \in (0, \frac{d+1}{2})$. It follows that

$$\lim_{k\to\infty}\widehat{g_{k,\alpha}}(\xi) = \frac{\pi^{\frac{d}{2}-\alpha}\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{d-\alpha}{2})}\,\frac{1}{|\xi|^\alpha}$$

for any $\xi \in \mathbb{R}^d \setminus \{0\}$, with $|\widehat{g_{k,\alpha}}(\xi)| \lesssim \frac{1}{|\xi|^\alpha}$ for any $\xi \in \mathbb{R}^d \setminus \{0\}$, uniformly in k. Let $\psi \in \mathcal{S}(\mathbb{R}^d)$. Observe that

$$|\widehat{g_{k,\alpha}}(\xi)\psi(\xi)| \lesssim \frac{|\psi(\xi)|}{|\xi|^\alpha}$$

uniformly in k, where $|\xi|^{-\alpha}|\psi(\xi)| \in L^1(\mathbb{R}^d)$ since $\alpha < n$ (recall that $\alpha < \frac{d+1}{2}$ and that $d \geq 2$) and $\psi \in \mathcal{S}(\mathbb{R}^d)$. Since $\widehat{g_{k,\alpha}}$ converges almost everywhere over $\mathbb{R}^d$, the dominated convergence theorem entails that

$$\int_{\mathbb{R}^d}\widehat{g_{k,\alpha}}(\xi)\psi(\xi)\,d\xi \to \frac{\pi^{\frac{d}{2}-\alpha}\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{d-\alpha}{2})}\int_{\mathbb{R}^d}\frac{1}{|\xi|^\alpha}\psi(\xi)\,d\xi$$

as $k \to \infty$. On the other hand, for any k, we have

$$\int_{\mathbb{R}^d}\widehat{g_{k,\alpha}}(\xi)\psi(\xi)\,d\xi = \int_{\mathbb{R}^d} g_{k,\alpha}(\xi)\widehat{\psi}(\xi)\,d\xi,$$

since $g_{k,\alpha} \in L^1(\mathbb{R}^d)$. Similarly, we have $|g_{k,\alpha}(x)\widehat{\psi}(x)| \leq |x|^{\alpha-n}|\psi(x)| \in L^1(\mathbb{R}^d)$, uniformly in k, and we have

$$\int_{\mathbb{R}^d} g_{k,\alpha}(\xi)\widehat{\psi}(\xi)\,d\xi \to \int_{\mathbb{R}^d}\frac{1}{|x|^{d-\alpha}}\widehat{\psi}(\xi)\,d\xi$$

by dominated convergence. It follows that

$$\forall\ \psi \in \mathcal{S}\mathbb{R}^d), \qquad \int_{\mathbb{R}^d}\frac{1}{|x|^{d-\alpha}}\widehat{\psi}(\xi)\,d\xi = \frac{\pi^{\frac{d}{2}-\alpha}\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{d-\alpha}{2})}\int_{\mathbb{R}^d}\frac{1}{|\xi|^\alpha}\psi(\xi)\,d\xi.$$

We deduce that

$$\mathcal{F}\Big(\frac{1}{|x|^{d-\alpha}}\Big)(\xi) = \frac{\pi^{\frac{d}{2}-\alpha}\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{d-\alpha}{2})}\frac{1}{|\xi|^\alpha} \tag{A.6}$$

in $\mathcal{S}(\mathbb{R}^d)'$ for any $\alpha \in (0, \frac{d+1}{2})$. Taking the inverse Fourier transform on both sides of (A.6) yields

$$\mathcal{F}\Big(\frac{1}{|x|^\alpha}\Big)(\xi) = \frac{\Gamma(\frac{d-\alpha}{2})}{\pi^{\frac{d}{2}-\alpha}\Gamma(\frac{\alpha}{2})}\frac{1}{|\xi|^{d-\alpha}} \tag{A.7}$$

in $\mathcal{S}(\mathbb{R}^d)'$ for any $\alpha \in (0, \frac{d+1}{2})$. Now let $\beta \in (\frac{d-1}{2}, d)$ and let us write $\beta = d - \alpha$ for some $\alpha \in (0, \frac{d+1}{2})$. Then, (A.6) yields

$$\mathcal{F}\Big(\frac{1}{|x|^\beta}\Big)(\xi) = \frac{\pi^{\frac{d}{2}-\alpha}\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{d-\alpha}{2})}\frac{1}{|\xi|^\alpha} = \frac{\Gamma(\frac{d-\beta}{2})}{\pi^{\frac{d}{2}-\beta}\Gamma(\frac{\beta}{2})}\frac{1}{|\xi|^{d-\beta}} \tag{A.8}$$

in $\mathcal{S}(\mathbb{R}^d)'$ for any $\beta \in (\frac{d-1}{2}, d)$. Putting (A.7) and (A.8) together yields the conclusion for any $\alpha \in (0, \frac{d+1}{2}) \cup (\frac{d-1}{2}, d) = (0, d)$. ■

The proof of Lemma A.3 requires the following result, that can be found in Appendix C.2 of [23].

Lemma A.2. *Let $\Omega \subset \mathbb{R}^d$ be a regular and bounded open subset, and $\partial\Omega$ denote its boundary. Let $u, v \in \mathcal{C}^1(\overline{\Omega})$. For any $i \in \{1, \ldots, n\}$, we have*

$$\int_\Omega (\partial_i u(x))v(x)\,dx = \int_{\partial\Omega} u(x)v(x)\nu_i(x)\,d\sigma(x) - \int_\Omega u(x)\partial_i v(x)\,dx,$$

where $\nu_i(x)$ is the ith component of the outer unit normal vector to Ω at x, and σ the surface area measure on $\partial\Omega$.

Lemma A.3. *The derivative of the tempered distribution $1/|x|^{d-1}$ is given by*

$$\nabla\Big(\frac{1}{|x|^{d-1}}\Big) = -(d-1)\ P.V.\Big(\frac{x}{|x|^{d+1}}\Big)$$

in $\mathcal{S}^d(\mathbb{R}^d)'$.

Proof of Lemma A.3. Let $\psi \in \mathcal{S}(\mathbb{R}^d)$. Because $x \mapsto 1/|x|^{d-1}$ is integrable near the origin in $\mathbb{R}^d$, Lebesgue's dominated convergence theorem yields

$$\begin{aligned}\Big\langle \nabla(1/|x|^{d-1}), \psi \Big\rangle &= -\Big\langle 1/|x|^{d-1}, \nabla\psi \Big\rangle \\ &= -\int_{\mathbb{R}^d} \frac{1}{|x|^{d-1}} \nabla\psi(x)\,dx \\ &= -\lim_{\substack{R\to\infty \\ \eta\to 0}} \int_{B_R\setminus B_\eta} \frac{1}{|x|^{d-1}}\nabla\psi(x)\,dx.\end{aligned}$$

Fix $0 < \eta < R < \infty$. Lemma A.2 entails that

$$\begin{aligned}\int_{B_R\setminus B_\eta} \frac{1}{|x|^{d-1}}\nabla\psi(x)\,dx &= -\int_{B_R\setminus B_\eta} \nabla\Big(\frac{1}{|x|^{d-1}}\Big)\psi(x)\,dx \\ &\quad + \int_{\partial B_R} \frac{1}{|x|^{d-1}}\psi(x)\frac{x}{|x|}\,d\sigma_R(x) - \int_{\partial B_\eta} \frac{1}{|x|^{d-1}}\psi(x)\frac{x}{|x|}\,d\sigma_\eta(x),\end{aligned}$$

where σ_r is the surface area measure on the sphere of radius r. Letting $u = x/R$, we find

$$\int_{\partial B_R} \frac{1}{|x|^{d-1}}\psi(x)\frac{x}{|x|}\,d\sigma_R(x) = R^{d-1}\int_{d-1} \frac{1}{R^{d-1}}\psi(Ru)u\,d\sigma_1(u).$$

Since ψ is bounded and $\psi(x) \to 0$ as $|x| \to \infty$, the latter converges to 0 as $R \to \infty$. Similarly, we have

$$\int_{\partial B_\eta} \frac{1}{|x|^{d-1}} \psi(x) \frac{x}{|x|} \, d\sigma_\eta(x) = \eta^{d-1} \int_{d-1} \frac{1}{\eta^{d-1}} \psi(\eta u) u \, d\sigma_1(u).$$

As $\eta \to 0$, the last integral converges to $\psi(0) \int_{d-1} u \, d\sigma_1(u) = 0$. It follows that

$$\begin{aligned}\left\langle \nabla(1/|x|^{d-1}), \psi \right\rangle &= \lim_{\substack{R\to\infty \\ \eta\to 0}} \int_{B_R \setminus B_\eta} \nabla\Big(\frac{1}{|x|^{d-1}}\Big) \psi(x) \, dx \\ &= -(d-1) \lim_{\substack{R\to\infty \\ \eta\to 0}} \int_{B_R \setminus B_\eta} \frac{x}{|x|^{d+1}} \psi(x) \, dx \\ &= -(d-1) \lim_{\eta \to 0} \int_{\mathbb{R}^d \setminus B_\eta} \frac{x}{|x|^{d+1}} \psi(x) \, dx \\ &= \left\langle -(d-1) \, \mathrm{P.V.}\Big(\frac{x}{|x|^{d+1}}\Big), \psi \right\rangle.\end{aligned}$$

This concludes the proof. ■

The following lemma, which we need to prove Theorem 3.3, is stated in [37], Corollary 2.2.10.

Lemma A.4. *Let Q and $(Q_k)_{k\geq 1}$ be Borel probability measures over $\mathbb{R}^d$ such that Q_k converges to Q in distribution as $k \to \infty$. Let $g : \mathbb{R}^d \to \mathbb{C}$ be a bounded and measurable map such that $Q(D_g) = 0$, where we let*

$$D_g := \{x \in \mathbb{R}^d : g \text{ is not continuous at } x\}.$$

Then $\int_{\mathbb{R}^d} g \, dQ_k \to \int_{\mathbb{R}^d} g \, dQ$ as $k \to \infty$.

Proof of Proposition 3.4. Assume that P is non-atomic over Ω. The continuity of R_P over Ω is a direct application of Lebesgue's dominated convergence theorem. Assume that P admits the density $f_P \in L^1(\Omega) \cap L^p_{\mathrm{loc}}(\Omega)$. We prove the result by induction. By the first part of the proof, we have $R_P \in \mathcal{C}^0(\Omega)$. In addition, we trivially have that $|R_P(x)| \leq 1$ for any $x \in \mathbb{R}^d$, so that $R_P \in \mathcal{C}^0_b(\mathbb{R}^d)$.

Let $0 \leq k \leq \ell - 1$ and assume that $R_P \in \mathcal{C}^k(\Omega)$ with

$$\partial^\alpha R_P(x) = \mathbb{E}[(\partial^\alpha K)(x - X)] \tag{A.9}$$

for any $x \in \Omega$ and $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq k$. Let us show that $R_P \in \mathcal{C}^{k+1}(\Omega)$ (and $R_P \in \mathcal{C}^{k+1}_b(\mathbb{R}^d)$ if $f_P \in L^p(\mathbb{R}^d)$) and that (A.9) holds for any $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq k+1$. For this purpose, let $\alpha \in \mathbb{N}^d$ with $|\alpha| = k$, $x \in \Omega$ and $r > 0$ be such that $\overline{B_r(x)} \subset \Omega$. Let $j \in \{1, \ldots, n\}$ and e_j be the jth vector of the canonical basis of $\mathbb{R}^d$. We are going to show that

$$\frac{\partial^\alpha R_P(x + he_j) - \partial^\alpha R_P(x)}{h} \to \mathbb{E}[(\partial_j \partial^\alpha K)(x - Z)]$$

$h \to 0$ and that the limit is continuous over Ω. Without loss of generality, let us assume that $|h| < \kappa$ for some $\kappa < d(x, \partial\Omega)$, so that $x + he_j \in B_\kappa(x) \subset \Omega$. For any h, let

$$S_h := \{x + she_j : s \in [0, 1]\}$$

be the line segment from x to $x + he_j$. Since $S_h \subset \Omega$ and P has a density over Ω, we have

$$\frac{\partial^\alpha R_P(x+he_j) - \partial^\alpha R_P(x)}{h} = \mathbb{E}\left[\frac{(\partial^\alpha K)(x+he_j - Z) - (\partial^\alpha K)(x-Z)}{h}\mathbb{I}[Z \in \mathbb{R}^d \setminus S_h]\right]$$

for any h. In order to take the limit as $h \to 0$ under the above expectation, we will show that the integrand is a uniformly P-integrable family indexed by h and converges P-almost surely as $h \to 0$. Since $K \in \mathcal{C}^\infty(\mathbb{R}^d \setminus \{0\})$, observe that

$$\frac{(\partial^\alpha K)(x+he_j - z) - (\partial^\alpha K)(x-z)}{h} = \int_0^1 (\partial_j \partial^\alpha K)(x + she_j - z)\, ds$$

for any $z \in \mathbb{R}^d \setminus S_h$. The latter obviously converges to $(\partial_j\partial^\alpha K)(x-z)$ as $h \to 0$, for any $z \in \mathbb{R}^d \setminus S_h$. Let us now show that the family of random vectors

$$\left(\int_0^1 (\partial_j \partial^\alpha K)(x + she_j - Z)\mathbb{I}[Z \in \mathbb{R}^d \setminus S_h]\, ds\right)_{|h|<\kappa}$$

is uniformly P-integrable. It is enough to show that there exists $\delta > 0$ such that

$$\sup_{|h|<\kappa} \mathbb{E}\left[\Big|\int_0^1 (\partial_j \partial^\alpha K)(x + she_j - Z)\, ds\Big|^{1+\delta}\mathbb{I}[Z \in \mathbb{R}^d \setminus S_h]\right] < \infty.$$

Let $\delta > 0$ be arbitrary for now and let us fix its value later on. Observe that $|\partial^\beta K(x)| \leq C_\beta |x|^{-|\beta|}$ for any $x \in \mathbb{R}^d \setminus \{0\}$, any $\beta \in \mathbb{N}^d$ and some positive constant C_β. Therefore, there exists $C > 0$ such that

$$\begin{aligned}
&\Big|\int_0^1 (\partial_j \partial^\alpha K)(x + she_j - z)\, ds\Big|^{1+\delta} \\
&\qquad \leq \int_0^1 |(\partial_j \partial^\alpha K)(x + she_j - z)|^{1+\delta}\, ds \\
&\qquad \leq C \int_0^1 \frac{1}{|x + she_j - z|^{(1+k)(1+\delta)}}\, ds
\end{aligned}$$

for any $z \in \mathbb{R}^d \setminus S_h$ and h, by Jensen's inequality. It follows from Fubini's theorem that

$$\begin{aligned}
&\sup_{|h|<\kappa} \mathbb{E}\left[\Big|\int_0^1 (\partial_j \partial^\alpha K)(x + she_j - Z)\, ds\Big|^{1+\delta}\mathbb{I}[Z \in \mathbb{R}^d \setminus S_h]\right] \\
&\qquad \leq C \sup_{|h|<\kappa} \int_0^1 \mathbb{E}\left[\frac{1}{|x + she_j - Z|^{(1+k)(1+\delta)}}\mathbb{I}[Z \in \mathbb{R}^d \setminus S_h]\right] ds \\
&\qquad \leq C \sup_{|h|<\kappa} \sup_{s\in[0,1]} \mathbb{E}\left[\frac{1}{|x + she_j - Z|^{(1+k)(1+\delta)}}\mathbb{I}[Z \in \mathbb{R}^d \setminus S_h]\right].
\end{aligned}$$

Let us fix h such that $|h| < \kappa$ and $s \in [0,1]$. We have that

$$\begin{aligned}
&\mathbb{E}\left[\frac{1}{|x + she_j - Z|^{(1+k)(1+\delta)}}\mathbb{I}[Z \in \mathbb{R}^d \setminus S_h]\right] \\
&\qquad \leq \frac{1}{r^{(1+k)(1+\delta)}} + \mathbb{E}\left[\frac{1}{|x + she_j - Z|^{(1+k)(1+\delta)}}\mathbb{I}[Z \in B_r(x+she_j) \setminus S_h]\right].
\end{aligned}$$

Since $B_r(x+she_j) \setminus S_h \subset \Omega$ and P admits a density f_Ω over Ω, Hölder's inequality yields

$$\begin{aligned}\mathbb{E}\bigg[&\frac{1}{|x+she_j-Z|^{(1+k)(1+\delta)}}\mathbb{I}[Z \in B_r(x+she_j)\setminus S_h]\bigg]\\ &= \int_{B_r(x+she_j)\setminus S_h} \frac{1}{|x+she_j-z|^{(1+k)(1+\delta)}} f_\Omega(z)\,dz\\ &\leq \bigg(\int_{B_r} \frac{1}{|z|^{q(1+k)(1+\delta)}}\,dz\bigg)^{1/q}\bigg(\int_{B_r(x+she_j)} |f_\Omega(z)|^p\,dz\bigg)^{1/p},\end{aligned}$$

where p is such that $f_\Omega \in L^p_{\text{loc}}(\Omega)$ and $q = \frac{p}{p-1}$ is the conjugate exponent of p. The fact that $k \leq \ell - 1$ and $p > \frac{d}{d-\ell}$ implies that $p > \frac{d}{d-(1+k)}$ and $q < \frac{d}{1+k}$. Let us therefore choose $\delta > 0$ small enough such that $q < \frac{d}{(1+k)(1+\delta)}$. In particular, we have

$$\int_{B_r} \frac{1}{|z|^{q(1+k)(1+\delta)}}\,dz < \infty.$$

Since $\overline{B_\kappa(x)} \subset \Omega$ and $f_\Omega \in L^p_{\text{loc}}(\Omega)$, we also have that

$$\int_{B_r(x+she_j)} |f_\Omega(z)|^p\,dz \leq \int_{B_\kappa(x)} |f_\Omega(z)|^p\,dz < \infty$$

uniformly in $|h| < \kappa$ and $s \in [0,1]$. We deduce that

$$\sup_{|h|<\kappa} \mathbb{E}\bigg[\Big|\int_0^1 (\partial_j\partial^\alpha K)(x+she_j-Z)\,ds\Big|^{1+\delta}\mathbb{I}[Z \in \mathbb{R}^d \setminus S_h]\bigg] < \infty.$$

Therefore, the family of random vectors

$$\bigg(\int_0^1 (\partial_j\partial^\alpha K)(x+she_j-Z)\mathbb{I}[Z \in \mathbb{R}^d \setminus S_h]\,ds\bigg)_{|h|<\kappa}$$

is uniformly P-integrable. It follows from Lebesgue-Vitali's theorem that

$$\frac{\partial^\alpha R_P(x+he_j) - \partial^\alpha R_P(x)}{h} \to \mathbb{E}[(\partial_j\partial^\alpha K)(x-Z)]$$

for any $x \in \Omega$ as $h \to 0$. Let us show that $x \mapsto \mathbb{E}[(\partial_j\partial^\alpha K)(x-Z)]$ is continuous over Ω. Let $x \in \Omega$ and $(x_m) \subset \Omega$ be a sequence converging to x as $m \to \infty$. The familiy of random vectors $((\partial_j\partial^\alpha K)(x_m - Z))_{d\in\mathbb{N}}$ converges P-almost surely to $(\partial_j\partial^\alpha K)(x-Z)$ as $m \to \infty$ since P is non-atomic, and is uniformly P-integrable since

$$\sup_{m\in\mathbb{N}} \mathbb{E}[|(\partial_j\partial^\alpha K)(x_m-Z)|^{1+\eta}] \lesssim \sup_{m\in\mathbb{N}} \mathbb{E}\bigg[\frac{1}{|x_m-Z|^{(1+k)(1+\eta)}}\bigg] < \infty$$

for η small enough, by the previous computations. It follows that $\partial^\alpha R_P \in \mathcal{C}^1(\Omega)$ and that

$$\partial_j\partial^\alpha R_P(x) = \mathbb{E}[(\partial_j\partial^\alpha K)(x-Z)]$$

for any $x \in \mathbb{R}^d$. Since $\alpha \in \mathbb{N}^d$ with $|\alpha| = k$ was arbitrary, we deduce that $R_P \in \mathcal{C}^{k+1}(\Omega)$ and that (A.9) holds for any $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq k+1$. We conclude by induction.

Assume that $\Omega = \mathbb{R}^d$, and let f_P stand for the density $f_\Omega \in L^p(\mathbb{R}^d)$. We showed that $R_P \in \mathcal{C}^\ell(\mathbb{R}^d)$. Then, fix $\alpha \in \mathbb{N}^d$ with $1 \le |\alpha| \le \ell$ and let us show that $\partial^\alpha R_P$ converges to 0 at infinity. For the sake of convenience, write $k := |\alpha|$. We have already noticed that

$$|\partial^\alpha R_P(x)| \le C\, \mathbb{E}\Big[\frac{1}{|x-Z|^k}\mathbb{I}[Z \neq x]\Big] =: C\, h(x)$$

for any $x \in \mathbb{R}^d$ and some positive constant C. Therefore, it is enough to show that h converges to 0 at infinity. Let $(x_m) \subset \mathbb{R}^d$ be such that $|x_m| \to \infty$ as $m \to \infty$. A standard application of Lebesgue's dominated convergence theorem entails that

$$\mathbb{E}\Big[\frac{1}{|x_m-Z|^k}\mathbb{I}[Z \in \mathbb{R}^d \setminus B_1(x)]\Big] \to 0$$

as $m \to \infty$. Next observe that

$$\begin{aligned}
&\Big|\mathbb{E}\Big[\frac{1}{|x_m-Z|^k}\mathbb{I}[B_1(x)]\Big]\Big| \\
&\quad = \int_{B_1(x_m)} \frac{1}{|z-x_m|^k} f_P(z)\, dz \\
&\quad \le C\Big(\int_{B_1(x_m)} |f_P(z)|^p\, dz\Big)^{1/p}\Big(\int_{B_1}\frac{1}{|z|^{qk}}\, dz\Big)^{1/q},
\end{aligned}$$

where $q = \frac{p}{p-1}$ is the conjugate exponent of p. Since $p > \frac{d}{d-\ell}$ and $k \le \ell$, we have $p > \frac{d}{d-k}$ whence $q < \frac{d}{k}$. In particular, $qk < n$. It follows that

$$\int_{B_1}\frac{1}{|z|^{qk}}\, dz < \infty.$$

It remains to show that $\int_{B_1(x_m)} |f_P(z)|^p\, dz \to 0$ as $m \to \infty$. Since $f_P \in L^p(\mathbb{R}^d)$, let ν be the non-negative finite measure defined by

$$\nu(B) := \int_B |f_P(z)|^p\, dz$$

for any Borel subset $B \subset \mathbb{R}^d$. We then have

$$\int_{B_1(x_m)} |f_P(z)|^p\, dz = \nu(B_1(x_m))$$

for any m. Furthermore, we have $\nu(\mathbb{R}^d \setminus B_{|x_m|-1}) \to 0$ as $m \to \infty$ since ν is finite and $|x_m| \to \infty$ as $m \to \infty$. It follows that

$$\mathbb{E}\Big[\frac{1}{|x_m-Z|^k}\mathbb{I}[B_1(x_m)]\Big] \to 0$$

as $m \to \infty$. We deduce that $\partial^\alpha R_P$ converges to 0 at infinity for any $\alpha \in \mathbb{N}^d$ with $1 \le |\alpha| \le \ell$, which concludes the proof. ∎

ECARES and Département de Mathématique, Université libre de Bruxelles
Email address: dimitri.konen@ulb.be